

Factorizations, Riemann-Hilbert problems and the corona theorem

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Abstract. The solvability of the Riemann-Hilbert boundary value problem on the real line is described in the case when its matrix coefficient admits a Wiener-Hopf type factorization with bounded outer factors but rather general diagonal elements of its middle factor. This covers, in particular, the almost periodic setting, when the factorization multiples belong to the algebra generated by the functions $e_\lambda(x) := e^{i\lambda x}$, $\lambda \in \mathbb{R}$. Connections with the corona problem are discussed. Based on those, constructive factorization criteria are derived for several types of triangular 2×2 matrices with diagonal entries $e_{\pm\lambda}$ and non-zero off diagonal entry of the form $a_-e_{-\beta} + a_+e_\nu$ with $\nu, \beta \geq 0$, $\nu + \beta > 0$ and a_\pm analytic and bounded in the upper/lower half plane.

1. Introduction

The (vector) Riemann-Hilbert boundary value problem on the real line \mathbb{R} consists in finding two vector functions ϕ_\pm , analytic in the upper and lower half plane $\mathbb{C}^\pm = \{z \in \mathbb{C} : \pm \operatorname{Im} z > 0\}$ respectively, satisfying the condition

$$\phi_- = G\phi_+ + g, \quad (1.1)$$

imposed on their boundary values on \mathbb{R} . Here g is a given vector function and G is a given matrix function defined on \mathbb{R} , of appropriate sizes. It is well known that

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various properties of (1.1) can be described in terms of the (right) *factorization* of its matrix coefficient G , that is, a representation of G as a product

$$G = G_- D G_+^{-1}, \quad (1.2)$$

where G_{\pm} and their inverses are analytic in \mathbb{C}^{\pm} and D is a diagonal matrix function with diagonal entries d_j of a certain prescribed structure. An exact definition of the factorization (1.2) is correlated with the setting of the problem (1.1), that is, the requirements on the boundary behavior of ϕ_{\pm} .

To introduce a specific example, denote by H_r^{\pm} the Hardy classes in \mathbb{C}^{\pm} and by L_r the Lebesgue space on \mathbb{R} , with $r \in (0, \infty]$. Let us also agree, for any set X , to denote by X^n ($X^{n \times n}$) the set of all n -vectors (respectively, $n \times n$ matrices) with entries in X .

With this notation at hand, recall that the L_p *setting* of (1.1) is the one for which $g \in L_p^n$ and $\phi_{\pm} \in (H_p^{\pm})^n$. An appropriate representation (1.2), in this setting with $p > 1$, is the so called L_p factorization of G : the representation (1.2) in which

$$\lambda_{\pm}^{-1} G_{\pm} \in (H_p^{\pm})^{n \times n}, \quad \lambda_{\pm}^{-1} G_{\pm}^{-1} \in (H_q^{\pm})^{n \times n} \text{ and } d_j = (\lambda_- / \lambda_+)^{\kappa_j}. \quad (1.3)$$

Here

$$1 < p < \infty, \quad q = \frac{p}{p-1}, \quad \lambda_{\pm}(z) = z \pm i,$$

and the integers κ_j are called the (right) *partial indices* of G .

A full solvability picture for the problem (1.1) with L_p factorable G can be extracted from [12, Chapter 3], see also [10]. The central result in this direction is (the real line version of) the Simonenko's theorem, according to which (1.1) has a unique solution for every right hand side g — equivalently, the associated Toeplitz operator $T_G =: P_+ G|_{(H_p^+)^n}$ is invertible — if and only if G admits an L_p factorization (1.2) with $D = I$, subject to the additional condition

$$G_- P_+ G_-^{-1} \text{ is a densely defined bounded operator on } L_p^n. \quad (1.4)$$

Here P_+ is the projection operator of L_p onto H_p^+ along H_p^- , defined on vector (or matrix) functions entrywise.

In this paper, we take particular interest in *bounded* factorizations for which in (1.2), by definition,

$$G_+^{\pm 1} \in (H_{\infty}^+)^{n \times n}, \quad G_-^{\pm 1} \in (H_{\infty}^-)^{n \times n}. \quad (1.5)$$

Of course, with d_j as in (1.3) a bounded factorization of G is its L_p factorization simultaneously for all $p \in (1, \infty)$, and the additional condition (1.4) is satisfied. However, some meaningful conclusions regarding the problem (1.1) can be drawn from the relation (1.2) satisfying (1.5) even without any additional information about the diagonal entries of D . This idea for L_p factorization on closed curves was first discussed in [13]; in Section 2 we give a detailed account of the bounded factorization version. That includes in particular the interplay between the factorization problem and the corona theorem.

Section 3 deals with the *almost periodic* (AP for short) setting, in which the elements of the matrix function involved belong to the algebra AP generated by the functions

$$e_\lambda(x) = e^{i\lambda x}, \quad \lambda \in \mathbb{R}, \quad (1.6)$$

the diagonal elements d_j being chosen among its generators e_λ . In this case not only we consider the solvability of (1.1) when G admits an AP factorization, but also address the converse question: what information on the existence and the properties of that factorization can be obtained from a solution to a homogeneous problem

$$G\phi_+ = \phi_-, \quad \phi_\pm \in (H_p^\pm)^n \quad (1.7)$$

with $p = \infty$.

In Sections 4, 5 we consider classes of matrix functions G for which (1.1) is closely related with a convolution equation on an interval of finite length. By determining a solution to the homogeneous Riemann-Hilbert problem (1.7) in H_∞^\pm and applying the results of the previous sections, we study the factorability of G and the properties of the related Toeplitz operator T_G . In particular, invertibility conditions for this operator are obtained and a subclass of matrix functions is identified for which invertibility of T_G is (somewhat surprisingly) equivalent to its semi-Fredholmness.

2. Riemann-Hilbert problems and factorization

We start with the description of the solutions to (1.1), in terms of a bounded factorization (1.2).

Theorem 2.1. *Let G admit a bounded factorization (1.2). Then all solutions of the problem (1.1) satisfying $\phi_\pm \in (H_p^\pm)^n$ for some $p \in [1, \infty]$ are given by*

$$\phi_+ = \sum_j \psi_j g_j^+, \quad \phi_- = \sum_j d_j \psi_j g_j^- + g. \quad (2.1)$$

Here g_j^\pm stands for the j -th column of G_\pm :

$$G_- = \begin{bmatrix} g_1^- & g_2^- & \cdots & g_n^- \end{bmatrix}, \quad G_+ = \begin{bmatrix} g_1^+ & g_2^+ & \cdots & g_n^+ \end{bmatrix}, \quad (2.2)$$

and ψ_j is an arbitrary function satisfying

$$\psi_j \in H_p^+, \quad d_j \psi_j + (G_-^{-1}g)_j \in H_p^-. \quad (2.3)$$

In other words, the Riemann-Hilbert problem (1.1) with a matrix coefficient G admitting a bounded factorization can be untangled into n scalar Riemann-Hilbert problems, in the same L_p setting.

The proof of Theorem 2.1 is standard in the factorization theory, based on a simple change of unknowns $\phi_\pm = G_\pm \psi_\pm$. We include it here for completeness.

Proof. If (ϕ_+, ϕ_-) is a solution to (1.1), then defining $\psi := (\psi_j)_{j=1, \dots, n} = G_+^{-1} \phi_+$ we get $\phi_+ = G_+ \psi$, $\phi_- = G_- D \psi + g$, which is equivalent to (2.1), and (2.3) is satisfied. Conversely, if (2.3) holds for all $j = 1, \dots, n$, then $\phi_+ = G_+ \psi \in (H_p^+)^n$, $\phi_- = G_- D \psi + g \in (H_p^-)^n$, and (1.1) holds. \square

We will say that a function f , defined a.e. on \mathbb{R} , is of *non-negative type* if

$$f \in H_\infty^+ \text{ or } f^{-1} \in H_\infty^-. \quad (2.4)$$

The type is *non-positive* if

$$f \in H_\infty^- \text{ or } f^{-1} \in H_\infty^+, \quad (2.5)$$

(strictly) *positive* if (2.4) holds while (2.5) does not, and *neutral* if both (2.4), (2.5) hold.

Lemma 2.2. *For d_j of positive type, there is at most one function ψ_j satisfying (2.3).*

Proof. It suffices to show that the only function $\psi \in H_p^+$ satisfying $d_j \psi \in H_p^-$ is zero.

If the first condition in (2.4) holds for $f = d_j$, then $d_j \psi \in H_p^+$ simultaneously with ψ itself. From here and $d_j \psi \in H_p^-$ it follows that $d_j \psi$ is a constant. If this constant is non-zero (which is only possible if $p = \infty$), then d_j is invertible in H_∞^+ which contradicts the strict positivity of its type. On the other hand, the product $d_j \psi$ of two analytic functions may be identically zero only if one of them is. It cannot be d_j (once again, since otherwise the first condition in (2.5) would hold); thus, $\psi = 0$.

The second case of (2.4) can be treated in a similar way. \square

As an immediate consequence we have:

Corollary 2.3. *If G admits a bounded factorization with all d_j of positive type, then the homogeneous Riemann-Hilbert problem (1.7) has only the trivial solution $\phi_+ = \phi_- = 0$ for any $p \in [1, \infty]$.*

If d_j is of neutral type, then by definition it is either invertible in H_∞^+ , or in H_∞^- , or is equal to zero. Disallowing the latter case, and absorbing d_j in the column g_j^\pm in the former, we may without loss of generality suppose that all such d_j are actually equal 1. With this convention in mind, the following result holds.

Corollary 2.4. *Let G admit a bounded factorization with all d_j of non-negative type, $d_j \neq 0$. Then the homogeneous problem (1.7) for $1 \leq p < \infty$ has only the trivial solution, and for $p = \infty$ all its solutions are given by*

$$\phi_+ = \sum_{j \in J} c_j g_j^+, \quad \phi_- = \sum_{j \in J} c_j g_j^-. \quad (2.6)$$

Here g_j^\pm are as in (2.2), $c_j \in \mathbb{C}$, and $j \in J$ if and only if d_j is of neutral type.

Proof. From (2.1) and from Lemma 2.2 we have

$$\phi_+ = \sum_{j \in J} \psi_j g_j^+, \quad \phi_- = \sum_{j \in J} d_j \psi_j g_j^-,$$

while our convention regarding the neutral type allows us to drop the functions d_j in the expression for ϕ_- . Finally, (2.3) with d_j of neutral type and $g = 0$ means that $\psi_j \in H_p^+ \cap H_p^-$, and thus ψ_j is a constant ($= 0$ if $p < \infty$). \square

Recall that the factorization (1.2) is *canonical* if the middle factor D of it is the identity matrix, and can therefore be dropped:

$$G = G_- G_+^{-1}. \quad (2.7)$$

The following criterion for bounded canonical factorability is easy to establish, and actually well known. We state it here, with proof, for the sake of completeness and ease of references.

Lemma 2.5. *G admits a bounded canonical factorization (2.7) if and only if problem (1.7) with $p = \infty$ has solutions ϕ_j^\pm , $j = 1, \dots, n$, such that*

$$\det[\phi_1^\pm \dots \phi_n^\pm] \text{ is invertible in } H_\infty^\pm. \quad (2.8)$$

If this is the case, then one of the factorizations is given by

$$G_\pm = [\phi_1^\pm \dots \phi_n^\pm], \quad (2.9)$$

and all solutions to (1.7) in H_∞^\pm are linear combinations of ϕ_j^\pm .

Proof. If (2.7) holds with G_\pm satisfying (1.5), then one may choose ϕ_j^\pm as the j -th column of G_\pm . Conversely, if ϕ_j^\pm satisfy (1.7) and (2.8), then G_\pm given by (2.9) satisfy $GG_+ = G_-$ and (1.5). Therefore, (2.7) holds and delivers a bounded canonical factorization of G .

The last statement now follows from Corollary 2.4. \square

Observe that for G with constant non-zero determinant, the determinants of matrix functions G_\pm given by (2.9) also are necessarily constant. So, (2.8) holds if and only if the vector functions $\phi_1^+(z), \dots, \phi_n^+(z)$ (or $\phi_1^-(z), \dots, \phi_n^-(z)$) are linearly independent for at least one value of $z \in \mathbb{C}^+$ (resp., \mathbb{C}^-).

As it happens, if G admits a bounded canonical factorization, all its bounded factorizations (with no a priori conditions on d_j) are forced to be “almost” canonical. The precise statement is as follows.

Theorem 2.6. *Let G have a bounded canonical factorization $G = \tilde{G}_- \tilde{G}_+^{-1}$. Then all its bounded factorizations are given by (1.2), where each d_j has a bounded canonical factorization*

$$d_j = d_{j-} d_{j+}^{-1}, \quad j = 1, \dots, n, \quad (2.10)$$

$$G_\pm = \tilde{G}_\pm Z D_\pm^{-1}, \quad D_\pm = \text{diag}[d_{1\pm}, \dots, d_{n\pm}], \quad (2.11)$$

and Z is an arbitrary invertible matrix in $\mathbb{C}^{n \times n}$.

Proof. Equating two factorizations $\tilde{G}_- \tilde{G}_+^{-1}$ and $G_- D G_+^{-1}$ yields

$$D = G_-^{-1} \tilde{G}_- \tilde{G}_+^{-1} G_+ = F_- F_+^{-1}, \quad (2.12)$$

where $F_\pm, F_\pm^{-1} \in (H_\infty^\pm)^{n \times n}$. Consequently, D admits a bounded canonical factorization, and therefore the Toeplitz operator T_D is invertible on $(H_p^+)^n$ for $p \in (1, \infty)$. Being the direct sum of n scalar Toeplitz operators T_{d_j} , this implies that each of the latter also is invertible, on H_p^+ . Thus, each of the scalar functions d_j admits a canonical L_p factorization. Let (2.10) be such a factorization, corresponding¹ to $p = 2$. Then, according to (2.12) the elements f_{ij}^\pm of the matrix functions F_\pm are related as $f_{ij}^- = d_j f_{ij}^+$. Due to the invertibility of F_\pm , for each j the functions f_{ij}^\pm are non-zero for at least one value of i . Choosing such i arbitrarily, and abbreviating the respective f_{ij}^\pm simply to $f_{j\pm}$, we have

$$f_{j-} d_{j-}^{-1} = f_{j+} d_{j+}^{-1}.$$

The left and right hand side of the latter equality is a function in $\lambda_- H_2^-$ and $\lambda_+ H_2^+$, respectively. Hence, each of them is just a scalar (non-zero, due to our choice of i). So, $d_{j\pm} \in H_\infty^\pm$.

Letting $d_\pm = \prod_{j=1}^n d_{j\pm}$, from here we obtain that $\det D = d_- d_+^{-1}$, with $d_\pm \in H_\infty^\pm$. But (2.12) implies also that $\det D$ admits the bounded analytic factorization $\det F_- / \det F_+$. Thus,

$$d_- / \det F_- = d_+ / \det F_+,$$

with left/right hand side lying in H_∞^\pm , respectively. Hence, d_\pm differs from $\det F_\pm$ only by a (clearly, non-zero) scalar multiple, and therefore is invertible in H_∞^\pm . This implies the invertibility of each multiple $d_{j\pm}$ in H_∞^\pm , $j = 1, \dots, n$, so that each representation (2.10) is in fact a bounded canonical factorization.

With the notation D_\pm as in (2.11), the first equality in (2.12) can be rewritten as

$$\tilde{G}_-^{-1} G_- D_- = \tilde{G}_+^{-1} G_+ D_+.$$

Since the left/right hand side is invertible in $(H_\infty^\mp)^{n \times n}$, each of them is in fact an invertible constant matrix Z . This implies the first formula in (2.11). \square

According to (2.11) with $D = I$, two bounded canonical factorizations of G are related as

$$G_\pm = \tilde{G}_\pm Z, \text{ where } Z \in \mathbb{C}^{n \times n}, \det Z \neq 0, \quad (2.13)$$

— a well-known fact.

When $n = 2$, the results proved above simplify in a natural way. We will state only one such simplification, once again, for convenience of references.

¹The interpolation property of factorization [12, Theorem 3.9] implies that in our setting the canonical L_p factorization of d_j is the same for all $p \in (1, \infty)$ but this fact has no impact on the reasoning.

Theorem 2.7. *Let G be a 2×2 matrix function admitting a bounded factorization (1.2) with one of the diagonal entries (say d_2) of positive type. Then the problem (1.7) has non-trivial solutions in H_p^\pm for some $p \in [1, \infty]$ if and only if d_1 admits a representation $d_1 = d_{1-}d_{1+}^{-1}$ with $d_{1\pm} \in H_p^\pm$. If this condition holds, then all the solutions of (1.7) are given by*

$$\phi_+ = \psi g_1^+, \quad \phi_- = d_1 \psi g_1^-,$$

where g_1^\pm is the first column of G_\pm in the factorization (1.2) and $\psi \in H_p^+$ is an arbitrary function satisfying $d_1 \psi \in H_p^-$.

Proof. Sufficiency. If $d_1 = d_{1-}d_{1+}^{-1}$ with $d_{1\pm} \in H_p^\pm$, then obviously $d_{1+} \neq 0$ and

$$\phi_+ = d_{1+}g_1^+, \quad \phi_- = d_{1-}g_1^-$$

is a non-trivial solution to (1.7).

Necessity. By Lemma 2.2 and Theorem 2.1 the solution must be of the form $\phi_+ = \psi g_1^+$, $\phi_- = d_1 \psi g_1^-$ with $\psi \in H_p^+ \setminus \{0\}$, $d_1 \psi \in H_p^-$. It remains to set $d_{1+} = \psi$, $d_{1-} = d_1 \psi$. \square

More interestingly, there is a close relation between factorization and corona problems.

Recall that a vector function ω with entries $\omega_1, \dots, \omega_n \in H_\infty^+$ satisfies the *corona condition* in \mathbb{C}^+ (notation: $\omega \in CP^+$) if and only if

$$\inf_{z \in \mathbb{C}^+} (|\omega_1(z)| + \dots + |\omega_n(z)|) > 0.$$

The *corona condition* in \mathbb{C}^- for a vector function $\omega \in (H_\infty^-)^n$ and the notation $\omega \in CP^-$ are introduced analogously.

By the corona theorem, $\omega \in CP^\pm$ if and only if there exists $\omega^* = (\omega_1^*, \dots, \omega_n^*) \in (H_\infty^\pm)^n$ such that

$$\omega_1 \omega_1^* + \dots + \omega_n \omega_n^* = 1.$$

Theorem 2.8. *If an $n \times n$ matrix function G admits a bounded canonical factorization, then any non-trivial solution of problem (1.7) in $(H_\infty^\pm)^n$ actually lies in CP^\pm .*

Proof. Let G admit a bounded canonical factorization (2.7). By Corollary 2.4, every non-trivial solution ϕ_\pm of (1.7) is a nontrivial linear combination of the columns g_j^\pm , $j = 1, \dots, n$. According to (2.13), any such combination, in turn, can be used as a column of some (perhaps, different) bounded canonical factorization of G . Being a column of an invertible element of $(H_\infty^\pm)^{n \times n}$, it must lie in CP^\pm . \square

The following result is a somewhat technical generalization of Theorem 2.8, which will be used later on.

Theorem 2.9. *Let G be an $n \times n$ matrix function admitting a bounded factorization (1.2) in which for all $k = 2, \dots, n$ either $d_k = d_1 \neq 0$ or $d_1^{-1}d_k$ is a function*

of positive type. Then for any pair of non-zero vector functions $\phi_{\pm} \in (H_{\infty}^{\pm})^n$ satisfying $G\phi_+ = \phi_-$, $d_1\phi_+ \in (H_{\infty}^+)^n$, in fact stronger conditions

$$d_1\phi_+ \in CP^+, \quad \phi_- \in CP^- \quad (2.14)$$

hold. In order for such pairs to exist, d_1 has to be of non-positive type.

Proof. Let $\tilde{G} = d_1^{-1}G$. Then, due to (1.2),

$$\tilde{G} = G_- \tilde{D} G_+^{-1} \text{ with } \tilde{D} = \text{diag}[1, d_1^{-1}d_2, \dots, d_1^{-1}d_n], \quad (2.15)$$

which of course is a bounded factorization of \tilde{G} .

Condition $\phi_- = G\phi_+$ implies that $\phi_- = \tilde{G}d_1\phi_+$, so that the pair $d_1\phi_+, \phi_-$ is a non-trivial solution of the homogeneous Riemann-Hilbert problem with the coefficient \tilde{G} .

If $d_1 = d_2 = \dots = d_n$, then (2.15) delivers a bounded canonical factorization of \tilde{G} , so that the desired result follows from Theorem 2.8. If, on the other hand, $d_1^{-1}d_2, \dots, d_1^{-1}d_n$ are all of positive type, then $d_1\phi_+$ and ϕ_- differ only by a (non-zero) constant scalar multiple from the first column of G_+ and G_- respectively, according to Corollary 2.4. This again implies (2.14).

Finally, from $d_1\phi_+ \in CP^+$ and $\phi_+ \in (H_{\infty}^+)^n$ it follows that $d_1^{-1} \in H_{\infty}^+$, that is, d_1 is of non-positive type. \square

The exact converse of Theorem 2.8 is not true. However, a slightly more subtle result holds.

Theorem 2.10. *Let $G \in L_{\infty}^{2 \times 2}$ be such that there exists a solution of problem (1.7) in CP^{\pm} . Then the Toeplitz operators T_G on $(H_p^+)^2$ and $T_{\det G}$ on H_p^+ are Fredholm only simultaneously, and their defect numbers coincide.*

Proof. The existence of the above mentioned solutions implies (see, e.g., computations in [3, Section 22.1]) that

$$G = X_- \begin{bmatrix} \det G & 0 \\ * & 1 \end{bmatrix} X_+,$$

where X_{\pm} is an invertible element of $(H_{\infty}^{\pm})^{2 \times 2}$. From here and elementary properties of block triangular operators it follows that the respective defect numbers (and thus the Fredholm behavior) of T_G and $T_{\det G}$ are the same. \square

According to Theorem 2.10, in the particular case when $\det G$ admits a canonical factorization, the operator T_G is invertible provided that (1.7) has a solution in CP^{\pm} . For $\det G \equiv 1$ the latter result was essentially established in [1]. An alternative, and more detailed, proof of Theorem 2.10 can be found in [5], Theorems 4.1 and 4.4.

Let now \mathcal{B} be a subalgebra of L_{∞} (not necessarily closed in L_{∞} norm) such that, for any n , a matrix function $G \in \mathcal{B}^{n \times n}$ admits a bounded canonical factorization if and only if the operator T_G is invertible in $(H_p^+)^n$ for at least one (and therefore all) $p \in (1, \infty)$. There are many classes satisfying this property, e.g.,

decomposable algebras of continuous functions (see [8, 12]) or the algebra APW considered below.

Theorem 2.11. *Let $G \in \mathcal{B}^{2 \times 2}$ with $\det G$ admitting a bounded canonical factorization, and let $\phi_{\pm} \in (H_{\infty}^{\pm})^2$ be a non-zero solution to (1.7). Then G has a bounded canonical factorization if and only if $\phi_{\pm} \in CP^{\pm}$.*

Proof. Necessity follows from Theorem 2.8 and sufficiency from Theorem 2.10. The latter can also be deduced from [1, Theorem 3.4] formulated there for G with constant determinant but remaining valid if $\det G$ merely admits a bounded canonical factorization. \square

3. AP factorization

We will now recast the results of the previous section in the framework of AP factorization. To this end, recall that AP is the uniform closure of all linear combinations $\sum c_j e_{\lambda_j}$, $c_j \in \mathbb{C}$, with e_{λ_j} defined by (1.6), while these linear combinations themselves form the set APP of all *almost periodic polynomials*. Properties of AP functions are discussed in detail in [9, 11], see also [3, Chapter 1]. In particular, for every $f \in AP$ there exists its *mean value*

$$M(f) = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T f(t) dt.$$

This yields the existence of $\hat{f}(\lambda) := M(e_{-\lambda} f)$, the *Bohr-Fourier coefficients* of f . For any given $f \in AP$, the set

$$\Omega(f) = \{\lambda \in \mathbb{R} : \hat{f}(\lambda) \neq 0\}$$

is at most countable, and is called the *Bohr-Fourier spectrum* of f . The formal *Bohr-Fourier series* $\sum_{\lambda \in \Omega(f)} \hat{f}(\lambda) e_{\lambda}$ may or may not converge; we will write $f \in APW$ if it does converge absolutely. The algebras AP and APW are inverse closed in L_{∞} ; moreover, for an invertible $f \in AP$ there exists an (obviously, unique) $\lambda \in \mathbb{R}$ such that a continuous branch of $\log(e_{-\lambda} f) \in AP$. This value of λ is called the *mean motion* of f ; we will denote it $\kappa(f)$.

Finally, let

$$AP^{\pm} = \{f \in AP : \Omega(f) \subset \mathbb{R}_{\pm}\},$$

where of course $\mathbb{R}_{\pm} = \{x \in \mathbb{R} : \pm x \geq 0\}$. Denote also

$$APW^{\pm} = AP^{\pm} \cap APW, \quad APP^{\pm} = AP^{\pm} \cap APP.$$

Clearly,

$$APP^{\pm} \subset APW^{\pm} \subset AP^{\pm} \subset H_{\infty}^{\pm}.$$

An AP factorization of G , by definition, is a representation (1.2) in which G_{\pm} are subject to the conditions

$$G_{+}^{\pm 1} \in (AP^{+})^{n \times n}, \quad G_{-}^{\pm 1} \in (AP^{-})^{n \times n}, \quad (3.1)$$

more restrictive than (1.5), and the diagonal entries of D are of the form $d_j = e_{\delta_j}$, $j = 1, \dots, n$. The real numbers δ_j are called the (right) partial AP indices of G , and by an obvious column permutation in G_{\pm} we may assume that they are arranged in a non-decreasing order: $\delta_1 \leq \delta_2 \leq \dots \leq \delta_n$.

A particular case of AP factorization occurs when conditions (3.1) are changed to more restrictive ones:

$$G_+^{\pm 1} \in (APW^+)^{n \times n}, \quad G_-^{\pm 1} \in (APW^-)^{n \times n},$$

or even

$$G_+^{\pm 1} \in (APP^+)^{n \times n}, \quad G_-^{\pm 1} \in (APP^-)^{n \times n}.$$

These are naturally called APW and APP factorization of G , respectively. Of course, G has to be an invertible element of $AP^{n \times n}$ ($APW^{n \times n}$, $APP^{n \times n}$) in order to admit an AP (resp., APW , APP) factorization. Moreover, the partial AP indices of G should then add up to the mean motion of its determinant:

$$\delta_1 + \dots + \delta_n = \kappa(\det G), \quad (3.2)$$

as can be seen by simply taking determinants of both sides.

All the statements of Section 2 are valid in these settings, and some of them can even be simplified. For instance, a diagonal element of D is of positive, negative or neutral type (in the sense of definitions (2.4), (2.5)) if and only if the corresponding partial AP index δ_j is respectively positive, negative or equal zero.

Corollary 2.4, for example, applies to AP -factorable matrix functions G with non-negative partial AP indices. Formulas (2.6) imply then that all solutions of (1.7) in $(H_{\infty}^{\pm})^n$ are automatically in $(AP^{\pm})^n$ (and even $(APW^{\pm})^n$ or $(APP^{\pm})^n$, provided that G is respectively APW - or APP -factorable).

Lemma 2.5 takes the following form.

Theorem 3.1. *An $n \times n$ matrix function G admits a canonical AP (APW) factorization if and only if there exist n solutions (ψ_j^+, ψ_j^-) to (1.7) in $(AP^{\pm})^n$ (resp., $(APW^{\pm})^n$), such that $\det[\psi_1^{\pm} \dots \psi_n^{\pm}]$ are bounded from zero in \mathbb{C}^{\pm} .*

The respective criterion for APP factorization is slightly different, because APP^{\pm} , as opposed to AP^{\pm} and APW^{\pm} , are not inverse closed in H_{∞}^{\pm} . Moreover, the only invertible elements of APP^{\pm} are non-zero constants. Therefore, we arrive at

Corollary 3.2. *An $n \times n$ matrix function G admits a canonical APP factorization if and only if there exist n solutions (ψ_j^+, ψ_j^-) to (1.7) in $(APP^{\pm})^n$ with constant non-zero $\det[\psi_1^{\pm} \dots \psi_n^{\pm}]$.*

Similarly to the case in Section 2, for matrix functions G with constant determinant the condition on $\det[\psi_1^{\pm} \dots \psi_n^{\pm}]$ holds whenever at least one of them is non-zero at just one point of $\mathbb{C}^{\pm} \cup \mathbb{R}$. All non-trivial solutions to (1.7) are actually in CP^{\pm} , as guaranteed by Theorem 2.8.

Theorem 2.1 of course remains valid when G admits an AP factorization; the only change needed is that d_j in formulas (2.1), (2.3) should be substituted by e_{δ_j} . For the homogeneous problem (1.7) this yields the following.

Theorem 3.3. *Let G admit an AP factorization (1.2). Then the general solution of problem (1.7) in $(H_\infty^\pm)^n$ is given by*

$$\phi_+ = \sum_j \psi_j g_j^+, \quad \phi_- = \sum_j e_{\delta_j} \psi_j g_j^-, \quad (3.3)$$

where the summation is with respect to those j for which $\delta_j \leq 0$, ψ_j are constant whenever $\delta_j = 0$ and satisfy

$$\psi_j \in H_\infty^+ \cap e_{-\delta_j} H_\infty^- \text{ whenever } \delta_j < 0. \quad (3.4)$$

Observe that ϕ_\pm given by (3.3) belong to AP^n if and only if condition (3.4) is replaced by a more restrictive

$$\psi_j \in AP, \quad \Omega(\psi_j) \subset [0, -\delta_j]$$

(where by convention $\psi_j = 0$ if $\delta_j > 0$), since

$$\psi := (\psi_j) = G_+^{-1} \phi_+ = D^{-1} G_-^{-1} \phi_-. \quad (3.5)$$

Moreover, if in fact G is APW factorable, then the functions (3.3) are in APW^n if and only if

$$\psi_j \in APW, \quad \Omega(\psi_j) \subset [0, -\delta_j].$$

Solutions of (1.7) in $(H_\infty^\pm)^n$ are automatically in AP (APW) if G is AP - (resp., APW -) factorable with non-negative partial AP indices, since in this case $D^{-1} \in APP^-$ and (3.5) implies that $\psi \in \mathbb{C}^n$. On the other hand, if G is APW factorable with at least one negative partial AP index, then all three classes are distinct. Indeed, for any j corresponding to $\delta_j < 0$ there is a plethora of functions ψ_j satisfying (3.4) not lying in AP , as well as functions in $AP \setminus APW$ with the Bohr-Fourier spectrum in $[0, -\delta_j]$.

The case of exactly one non-positive partial AP index is of special interest.

Corollary 3.4. *Let G admit an AP factorization with the partial AP indices $\delta_1 \leq 0 < \delta_2 \leq \dots$. Then all solutions to (1.7) in $(H_\infty^+)^n$ (AP^n , APW^n) are given by*

$$\phi_+ = f g_1^+, \quad \phi_- = e_{\delta_1} f g_1^-, \quad (3.6)$$

where f is an arbitrary H_∞^+ function such that $e_{\delta_1} f \in H_\infty^-$ (resp., $f \in AP$ or $f \in APW$ and $\Omega(f) \subset [0, -\delta_1]$).

For $n = 2$ the reasoning of Theorem 2.9 suggests an appropriate modification of (1.7) for which some solutions are forced to lie in AP . Recall our convention $\delta_1 \leq \delta_2$ according to which the condition on d_1, d_2 in Theorem 2.9 holds automatically.

Theorem 3.5. *Let G be a 2×2 AP factorable matrix function with partial indices δ_1, δ_2 ($\delta_1 \leq \delta_2$). Then any non-zero pair (ϕ_+, ϕ_-) with $\phi_+ \in (H_\infty^+)^2 \cap e_{-\delta_1}(H_\infty^+)^2$, $\phi_- = G\phi_+ \in (H_\infty^-)^2$ satisfies*

$$\phi_\pm \in (AP^\pm)^2, \quad e_{\delta_1}\phi_+ \in CP^+, \quad \phi_- \in CP^-,$$

and in order for such pairs to exist it is necessary and sufficient that $\delta_1 \leq 0$. If $\delta_2 > \delta_1$, all those solutions have the form

$$\phi_+ = ce_{-\delta_1}g_1^+, \quad \phi_- = cg_1^- \quad \text{with } c \in \mathbb{C} \setminus \{0\}.$$

For $\delta_2 = \delta_1$, ϕ_+ and ϕ_- are the same non-trivial linear combinations of the columns of $e_{-\delta_1}G_+$ and G_- .

Of course, Theorem 3.5 holds with AP changed to APW or APP everywhere in its statement.

Recall that a Toeplitz operator with scalar AP symbol f is Fredholm on H_p^+ for some (equivalently: all) $p \in (1, \infty)$ if and only if it is invertible if and only if f is invertible in AP with mean motion zero. Therefore, Theorem 2.10 implies

Lemma 3.6. *Let $G \in AP^{2 \times 2}$ be such that there exists a solution of (1.7) in CP^\pm . Then the Toeplitz operator T_G is invertible on $(H_p^+)^2$, $1 < p < \infty$, if and only if $\kappa(\det G) = 0$.*

Passing to the APW setting, we invoke the result according to which T_G with $G \in APW^{n \times n}$ is invertible if and only if G admits a canonical AP (or APW) factorization. Lemma 3.6 then implies (compare with Theorem 2.11):

Theorem 3.7. *Let $G \in APW^{2 \times 2}$. Then G admits a canonical AP factorization if and only if $\kappa(\det G) = 0$ and problem (1.7) has a solution in CP^\pm . If this is the case, then every non-zero solution of (1.7) is in $(APW^\pm)^2 \cap CP^\pm$.*

The first part of Theorem 3.7 for G with $\det G \equiv 1$ (so that $\kappa(\det G) = 0$ automatically) is in [3] (see Theorem 23.1 there). Essentially, it was proved in [1], with sufficiency following from Theorems 3.4, 6.1 and necessity from Theorem 3.5 there.

Our next goal is the APW factorization criterion in the not necessarily canonical case.

Theorem 3.8. *Let G be a 2×2 invertible APW matrix function. Denote $\delta = \kappa(\det G)$. Then G admits an APW factorization if and only if the Riemann-Hilbert problem*

$$e_{-\frac{\delta}{2}}G\psi_+ = \psi_-, \quad \psi_\pm \in (APW^\pm)^2 \tag{3.7}$$

admits a solution (ψ_+, ψ_-) such that

$$\tilde{\psi}_+ := e_{-\tilde{\delta}}\psi_+ \in CP^+ \quad \text{for some } \tilde{\delta} \geq 0 \quad \text{and } \psi_- \in CP^-. \tag{3.8}$$

If this is the case, then the partial AP indices of G are $\delta_1 = -\tilde{\delta} + \frac{\delta}{2}$, $\delta_2 = \tilde{\delta} + \frac{\delta}{2}$ and the factors G_\pm can be chosen in such a way that $\tilde{\psi}_+$ is the first column of G_+ and ψ_- is the first column of G_- .

Proof. If G admits an APW factorization, then $\delta = \delta_1 + \delta_2$ due to (3.2). In its turn, $\psi_+ = e_{\frac{\delta}{2}-\delta_1} g_1^+$, $\psi_- = g_1^-$ is a solution of (3.7) if $\frac{\delta}{2} - \delta_1 \geq 0$. It remains to set $\tilde{\delta} = \frac{\delta}{2} - \delta_1$ in order to satisfy (3.8) by analogy with Theorem 3.5. Formulas $\delta_1 = \frac{\delta}{2} - \tilde{\delta}$, $\delta_2 = \frac{\delta}{2} + \tilde{\delta}$ for the partial AP indices then also hold.

Suppose now that (3.7) has a solution for which (3.8) holds. From the corona theorem in the APW setting (see [3, Chapter 12]), there exist $h_{\pm} = (h_{1\pm}, h_{2\pm}) \in (APW^{\pm})^2$ such that

$$\psi_{1-} h_{1-} + \psi_{2-} h_{2-} = 1, \quad e_{-\tilde{\delta}}(\psi_{1+} h_{1+} + \psi_{2+} h_{2+}) = 1. \quad (3.9)$$

In other words, the matrix functions

$$H_+ = \begin{bmatrix} e_{-\tilde{\delta}} \psi_{1+} & -h_{2+} \\ e_{-\tilde{\delta}} \psi_{2+} & h_{1+} \end{bmatrix} \quad \text{and} \quad H_- = \begin{bmatrix} \psi_{1-} & -h_{2-} \\ \psi_{2-} & h_{1-} \end{bmatrix} \quad (3.10)$$

have determinants equal to 1 and are therefore invertible in $(APW^+)^{2 \times 2}$ and $(APW^-)^{2 \times 2}$ respectively. Thus the matrix functions $G_1 = H_-^{-1} G H_+$ and G are only simultaneously APW factorable, and their partial AP indices coincide.

For the first column of G_1 , taking (3.9) into account, we have

$$e_{-\tilde{\delta}} H_-^{-1} G \psi_+ = e_{\frac{\delta}{2}-\tilde{\delta}} H_-^{-1} \psi_- = \begin{bmatrix} e_{\frac{\delta}{2}-\tilde{\delta}} \\ 0 \end{bmatrix}.$$

Thus the second diagonal entry in G_1 must be equal to

$$e_{\tilde{\delta}-\frac{\delta}{2}} \det G = \gamma_- e_{\tilde{\delta}+\frac{\delta}{2}} \gamma_+^{-1},$$

where

$$\det G = \gamma_- e_{\delta} \gamma_+^{-1}$$

is a factorization of the scalar APW function $\det G$. Consequently,

$$G_1 = \begin{bmatrix} 1 & 0 \\ 0 & \gamma_- \end{bmatrix} \begin{bmatrix} e_{\frac{\delta}{2}-\tilde{\delta}} & g \\ 0 & e_{\frac{\delta}{2}+\tilde{\delta}} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \gamma_+ \end{bmatrix}^{-1} \quad (3.11)$$

with $g \in APW$ given by $g = [1 \ 0] G_1 [0 \ \gamma_+]^T$. Finally, the middle factor in the right-hand side of (3.11) is APW factorable with the partial indices $\frac{\delta}{2} - \tilde{\delta}$, $\frac{\delta}{2} + \tilde{\delta}$ equal to the mean motions of its diagonal entries:

$$\begin{bmatrix} e_{\frac{\delta}{2}-\tilde{\delta}} & g \\ 0 & e_{\frac{\delta}{2}+\tilde{\delta}} \end{bmatrix} = \begin{bmatrix} 1 & g_- \\ 0 & 1 \end{bmatrix} \begin{bmatrix} e_{\frac{\delta}{2}-\tilde{\delta}} & 0 \\ 0 & e_{\frac{\delta}{2}+\tilde{\delta}} \end{bmatrix} \begin{bmatrix} 1 & -g_+ \\ 0 & 1 \end{bmatrix}^{-1}. \quad (3.12)$$

The only condition on $g_{\pm} \in APW^{\pm}$ is

$$g e_{\tilde{\delta}-\frac{\delta}{2}} = g_+ + g_- e_{2\tilde{\delta}}, \quad (3.13)$$

and it can be satisfied since $\tilde{\delta} \geq 0$. Clearly, making use of (2.13) we can always choose G_{\pm} in such a way that $\tilde{\psi}_+$ is the first column of G_+ and ψ_- is the first column of G_- . \square

The proof of the preceding theorem provides, via (3.10), (3.11)–(3.13), formulas for an *APW* factorization of $G = H_-^{-1}G_1H_+$, in terms of the solutions to (3.7) and the corona problems (3.9).

4. Applications to a class of matrices with a spectral gap near zero

We consider now the factorability problem for a class of triangular matrix functions, closely related to the study of convolution equations on an interval of finite length λ (see, e.g., [3, Section 1.7] and references therein), of the form

$$G = \begin{bmatrix} e_{-\lambda} & 0 \\ g & e_\lambda \end{bmatrix}. \quad (4.1)$$

Throughout this section we assume that

$$g = a_-e_{-\beta} + a_+e_\nu \text{ for some } a_\pm \in H_\infty^\pm \text{ and } 0 \leq \nu, \beta \leq \lambda, \nu + \beta > 0. \quad (4.2)$$

Representation (4.2), when it exists, is not unique. In particular, it can be rewritten as

$$g = \tilde{a}_-e_{-\tilde{\beta}} + \tilde{a}_+e_{\tilde{\nu}}$$

with

$$\tilde{\nu} \in [0, \nu], \tilde{\beta} \in [0, \beta], \tilde{a}_+ = a_+e_{\nu-\tilde{\nu}}, \tilde{a}_- = a_-e_{\tilde{\beta}-\beta}. \quad (4.3)$$

Among all the representations (4.2) choose those with the smallest possible value of

$$N = \left\lceil \frac{\lambda}{\nu + \beta} \right\rceil, \quad (4.4)$$

where as usual $\lceil x \rceil$ denotes the smallest integer which is greater or equal to $x \in \mathbb{R}$. Of course, $N \geq 1$ due to the positivity of $\frac{\lambda}{\nu+\beta}$.

Formula (4.4) means that

$$N - 1 < \frac{\lambda}{\nu + \beta} \leq N.$$

Decreasing β, ν as described in (4.3), we may turn the last inequality into an equality. In other words, without loss of generality we may (and will) suppose that

$$\frac{\lambda}{\nu + \beta} = N \quad (4.5)$$

is an integer.

We remark that even under condition (4.5) representation (4.2) may not be defined uniquely.

Given $N \geq 1$, we denote by $S_{\lambda, N}$ the class of functions g satisfying (4.2), (4.5) for which

$$b_+ := e_{\frac{\beta}{N-1}}a_- \in H_\infty^+, \quad b_- := e_{-\frac{\nu}{N-1}}a_+ \in H_\infty^- \text{ if } N > 1. \quad (4.6)$$

By $\mathfrak{S}_{\lambda, N}$ we denote the class of 2×2 matrix functions G of the form (4.1) with $g \in S_{\lambda, N}$.

Remark 4.1. If $g \in S_{\lambda,N}$ with $N > 1$, then necessarily in (4.2) $\beta, \nu > 0$. Indeed, if say $\nu = 0$, then (4.6) implies that a_+ is a constant. Consequently, $g \in H_\infty^-$, and setting $a_- = g$, $a_+ = 0$, $\beta = 0$, $\nu = \lambda$ in (4.2) would yield $N = 1$ — a contradiction with our convention to choose the smallest possible value of N . Note also that, due to (4.6), a_\pm are entire functions when $N > 1$.

We start by determining a solution to (1.7) for G in $\mathfrak{S}_{\lambda,N}$.

Theorem 4.2. Let $G \in \mathfrak{S}_{\lambda,N}$, with g given by (4.2). Then

$$\phi_{1+} = e_{\lambda-\nu} \sum_{j=0}^{N-1} \left((-1)^j a_+^{N-1-j} a_-^j e_{-j\frac{\lambda}{N}} \right) \quad , \quad \phi_{2+} = -a_+^N, \quad (4.7)$$

$$\phi_{1-} = e_{-\lambda} \phi_{1+} \quad , \quad \phi_{2-} = (-1)^{N-1} a_-^N \quad (4.8)$$

deliver a solution $\phi_\pm = (\phi_{1\pm}, \phi_{2\pm})$ to the Riemann-Hilbert problem (1.7).

Proof. A direct computation based on the equality

$$x^N + (-1)^{N-1} y^N = (x+y) \sum_{j=0}^{N-1} ((-1)^j x^{N-1-j} y^j)$$

shows that $G\phi_+ = \phi_-$. Obviously, $\phi_{2\pm} \in H_\infty^\pm$. So, it remains to prove only that $\phi_{1\pm} \in H_\infty^\pm$. For $N = 1$, this is true because the definition of ϕ_{1+} from (4.7) collapses to $\phi_{1+} = e_\beta$. The case $N > 1$ is slightly more involved.

Namely, for $N > 1$ from (4.6) it follows that

$$e_{\frac{\beta}{N-1}} a_- = b_+ \in H_\infty^+,$$

so that

$$\phi_{1+} = \sum_{j=0}^{N-1} \left((-1)^j a_+^{N-1-j} b_+^j e_{\beta-j\frac{\beta}{N-1}} e_{\lambda-(j+1)\frac{\lambda}{N}} \right) \in H_\infty^+. \quad (4.9)$$

Analogously, from

$$e_{-\frac{\nu}{N-1}} a_+ = b_- \in H_\infty^-$$

we have

$$\phi_{1-} = \sum_{j=0}^{N-1} \left((-1)^j b_-^{N-1-j} a_-^j e_{-j(\frac{\nu}{N-1} + \frac{\lambda}{N})} \right) \in H_\infty^-. \quad (4.10)$$

□

This theorem, along with Theorem 2.10, allows to establish sufficient conditions, which in some cases are also necessary, for invertibility in $(H_p^+)^2$, $p > 1$, of Toeplitz operators with symbol $G \in \mathfrak{S}_{\lambda,N}$. To invoke Theorem 2.10, however, we need to be able to check when the pairs $(\phi_{1\pm}, \phi_{2\pm})$ defined by (4.7), (4.8) belong to CP^+ or CP^- . The following result from [4] (see Theorem 2.3 there) will simplify this task.

Theorem 4.3. *Let a 2×2 matrix function G and its inverse G^{-1} be analytic and bounded in a strip*

$$S = \{\xi \in \mathbb{C} : -\varepsilon_2 < \operatorname{Im} \xi < \varepsilon_1\} \quad \text{with} \quad \varepsilon_1, \varepsilon_2 \in [0, +\infty[, \quad (4.11)$$

and let $\phi_{\pm} \in (H_{\infty}^{\pm})^2$ satisfy (1.7). Then $\phi_+ \in CP^+$ (resp. $\phi_- \in CP^-$) if and only if

$$\inf_{\mathbb{C}^+ + i\varepsilon_1} (|\phi_{1+}| + |\phi_{2+}|) > 0 \quad \left(\text{resp.,} \quad \inf_{\mathbb{C}^- - i\varepsilon_2} (|\phi_{1-}| + |\phi_{2-}|) > 0 \right) \quad (4.12)$$

and one of the following (equivalent) conditions is satisfied:

$$\inf_S (|\phi_{1+}| + |\phi_{2+}|) > 0, \quad (4.13)$$

$$\inf_S (|\phi_{1-}| + |\phi_{2-}|) > 0. \quad (4.14)$$

Here and in what follows, we identify the functions ϕ_{1+}, ϕ_{2+} (resp., ϕ_{1-}, ϕ_{2-}) with their analytic extensions to $\mathbb{C}^+ - i\varepsilon_2$ (resp. $\mathbb{C}^+ + i\varepsilon_1$) and, for any real-valued function ϕ defined on S , abbreviate $\inf_{\zeta \in S} \phi(\zeta)$ to $\inf_S \phi$.

We will see that for $G \in \mathfrak{S}_{\lambda, N}$, $N \geq 1$, the behavior of the solutions “at infinity”, that is, condition (4.12) for sufficiently big $\varepsilon_1, \varepsilon_2 > 0$, is not difficult to study. Therefore, due to Theorem 4.3, we will be left with studying the behavior of ϕ_+ or ϕ_- in a strip of the complex plane. According to the next result this, in turn, can be done in term of the functions a_{\pm} from (4.2) or, equivalently, of g_{\pm} defined by

$$g_+ = e_{\nu} a_+, \quad g_- = e_{\nu - \frac{\lambda}{N}} a_-$$

It should be noted that, for $N > 1$, a_{\pm} and g_{\pm} are entire functions. Moreover, even if the behaviour of a_+ and a_- in a strip S may be difficult to study, it is clear from (4.7) and (4.8) that this is in general a much simpler task than that of checking whether (4.12) is satisfied using the expressions for $\phi_{1\pm}, \phi_{2\pm}$.

Lemma 4.4. *Let $G \in \mathfrak{S}_{\lambda, N}$ for some $N > 1$, and let ϕ_{\pm} be given by (4.7), (4.8). Then for any strip (4.11) we have*

$$\inf_S (|\phi_{1+}| + |\phi_{2+}|) > 0 \iff \inf_S (|a_+| + |a_-|) > 0 \iff \inf_S (|g_+| + |g_-|) > 0. \quad (4.15)$$

Proof. Since the last two conditions in (4.15) are obviously equivalent, and (4.13) is equivalent to (4.14) due to Theorem 4.3, we need to prove only that

$$\inf_S (|\phi_{1+}| + |\phi_{2+}|) > 0 \iff \inf_S (|a_+| + |a_-|) > 0.$$

Suppose first that

$$\inf_{\xi \in S} (|a_+(\xi)| + |a_-(\xi)|) = 0.$$

Then there is a sequence $\{\xi_n\}_{n \in \mathbb{N}}$ with $\xi_n \in S$ such that $a_+(\xi_n) \rightarrow 0$ and $a_-(\xi_n) \rightarrow 0$. Taking into account the expressions for ϕ_{1+}, ϕ_{2+} given by (4.7), we must have $\phi_{1+}(\xi_n) \rightarrow 0$ and $\phi_{2+}(\xi_n) \rightarrow 0$. Therefore,

$$\inf_{\xi \in S} (|\phi_{1+}(\xi)| + |\phi_{2+}(\xi)|) = 0.$$

Conversely, if

$$\inf_{\xi \in S} (|\phi_{1+}(\xi)| + |\phi_{2+}(\xi)|) = 0,$$

then for some sequence $\{\xi_n\}$ with $\xi_n \in S$ for all $n \in \mathbb{N}$, we have $\phi_{1+}(\xi_n) \rightarrow 0$ and $\phi_{2+}(\xi_n) \rightarrow 0$. Thus, from the expression for ϕ_{2+} given by (4.7), it follows that $a_+(\xi_n) \rightarrow 0$. From the expression for ϕ_{1+} in (4.7), we then conclude

$$a_-^{N-1} = (-1)^{N-1} e_{\nu-\frac{\lambda}{N}} \phi_{1+} + (-1)^N e_{\frac{N-1}{N}\lambda} \sum_{j=0}^{N-2} \left((-1)^j a_+^{N-1-j} a_-^j e_{-j\frac{\lambda}{N}} \right).$$

Since $\phi_{1+}(\xi_n) \rightarrow 0$ and $a_+(\xi_n) \rightarrow 0$, then also $a_-(\xi_n) \rightarrow 0$ and therefore

$$\inf_{\xi \in S} (|a_+(\xi)| + |a_-(\xi)|) = 0.$$

□

We can now state the following.

Theorem 4.5. *Let $G \in \mathfrak{S}_{\lambda,N}$ for some $N \in \mathbb{N}$, and let ϕ_{\pm} be the solutions to (1.7) given by (4.7), (4.8). Then:*

(i): *For $N = 1$, $\phi_{\pm} \in CP^{\pm}$ if and only if*

$$\inf_{\mathbb{C}^+ + i\varepsilon_1} |a_+| > 0, \quad \inf_{\mathbb{C}^- - i\varepsilon_2} |a_-| > 0 \text{ for some } \varepsilon_1, \varepsilon_2 > 0. \quad (4.16)$$

(ii): *For $N > 1$, $\phi_{\pm} \in CP^{\pm}$ if and only if, with b_+, b_- defined by (4.6),*

$$\inf_{\mathbb{C}^+ + i\varepsilon_1} (|b_+| + |a_+|) > 0, \quad \inf_{\mathbb{C}^- - i\varepsilon_2} (|b_-| + |a_-|) > 0 \text{ for some } \varepsilon_1, \varepsilon_2 > 0 \quad (4.17)$$

and, for any S of the form (4.11),

$$\inf_S (|a_+| + |a_-|) > 0. \quad (4.18)$$

Proof. Part (i) follows immediately from the explicit formulas

$$\phi_+ = (e_{\beta}, -a_+), \quad \phi_- = (e_{-\nu}, a_-). \quad (4.19)$$

(ii) For $N > 1$ we have, from (4.7)–(4.10),

$$\phi_{1+} = (-1)^{N-1} b_+^{N-1} + \sum_{j=0}^{N-2} \left((-1)^j a_+^{N-1-j} b_+^j e_{(N-1-j)\frac{\lambda-\nu}{N-1}} \right), \quad \phi_{2+} = (-1)^N a_+^N \quad (4.20)$$

$$\phi_{1-} = b_-^{N-1} + \sum_{j=1}^{N-1} \left((-1)^j a_-^j b_-^{N-1-j} e_{-j\frac{\lambda-\beta}{N-1}} \right), \quad \phi_{2-} = (-1)^{N-1} a_-^N. \quad (4.21)$$

Since $\nu, \beta < \lambda$ when $N > 1$, we see that for any sequence $\{\xi_n\}$ with $\xi_n \in \mathbb{C}^+$ and $\text{Im}(\xi_n) \rightarrow +\infty$,

$$|\phi_{1+} - (-1)^{N-1} b_+^{N-1}|_{(\xi_n)} \rightarrow 0, \quad (4.22)$$

and, for any sequence $\{\xi_n\}$ with $\xi_n \in \mathbb{C}^-$ and $\text{Im}(\xi_n) \rightarrow -\infty$,

$$|\phi_{1-} - b_-^{N-1}|_{(\xi_n)} \rightarrow 0. \quad (4.23)$$

It follows from (4.20)–(4.23) that there exist $\varepsilon_1, \varepsilon_2 > 0$ such that (4.17) holds if and only if there exist $\varepsilon_1, \varepsilon_2 > 0$ such that (4.12) holds. Moreover, by Lemma 4.4, (4.18) is equivalent to (4.13), thus the result follows from Theorem 4.3. \square

Note that $\det G \equiv 1$ for all matrix functions of the form (4.1). Therefore, Theorems 2.10, 2.11 and 4.5 combined imply the following.

Corollary 4.6. *Let the assumptions of Theorem 4.5 hold. Then condition (4.16) (for $N = 1$) and (4.17), (4.18) (for $N > 1$) imply the invertibility of T_G . The converse is also true (and, moreover, G admits a bounded canonical factorization) provided that $G \in \mathcal{B}^{2 \times 2}$.*

For $N = 1$, this result was proved (assuming $\lambda = 1$, which amounts to a simple change of variable) in [6], Theorem 4.1 and Corollary 4.5.

For the particular case when a_- (or a_+) is just a single exponential function, condition (4.18) is always satisfied and we can go deeper in the study of the properties of T_G . Before proceeding in this direction, however, it is useful to establish a more explicit characterization of the classes $S_{\lambda, N}$ under the circumstances. Without loss of generality, let us concentrate on the case when a_- is an exponential.

Lemma 4.7. *Given $\lambda > 0$, let*

$$g = e_{-\sigma} + g_+, \quad (4.24)$$

where $g_+ \in H_\infty^+$ is not identically zero, and $0 < \sigma < \lambda$. Then $g \in S_{\lambda, N}$ for some $N \in \mathbb{N}$ if and only if

$$e_{-\nu} g_+ \in H_\infty^+, \quad e_{-\frac{N}{N-1}\nu} g_+ \in H_\infty^- \quad (4.25)$$

for some

$$\nu \in \left[\frac{\lambda}{N} - \sigma, \frac{\lambda}{N} - \frac{N-1}{N}\sigma \right] \quad (4.26)$$

(of course, the second condition in (4.25) applies only for $N > 1$).

Note that conditions (4.25), (4.26) imply

$$e_{-\frac{\lambda}{N} + \sigma} g_+ \in H_\infty^+, \quad e_{-\frac{\lambda}{N-1} + \sigma} g_+ \in H_\infty^-,$$

and therefore may hold for at most one value of N .

Proof. Necessity. Suppose $g \in S_{\lambda, N}$. Comparing (4.2) and (4.24) we see that

$$a_- = e_{\beta - \sigma} \in H_\infty^-, \quad a_+ = e_{-\nu} g_+ \in H_\infty^+. \quad (4.27)$$

On the other hand, (4.6) takes the form

$$e_{\frac{N}{N-1}\beta - \sigma} \in H_\infty^+, \quad e_{-\frac{N}{N-1}\nu} g_+ \in H_\infty^-. \quad (4.28)$$

The first containments in (4.27), (4.28) are equivalent to

$$\frac{N-1}{N}\sigma \leq \beta \leq \sigma,$$

which along with (4.5) yields that $\nu = \frac{\lambda}{N} - \beta$ satisfies (4.26). The second containments in (4.27), (4.28) then imply (4.25).

Sufficiency. Given (4.25), (4.26), let $\beta = \frac{\lambda}{N} - \nu$, and define a_{\pm} by (4.27). Then (4.2), (4.5) and (4.6) hold (the latter for $N > 1$). \square

Theorem 4.8. *Let G be given by (4.1) with g of the form*

$$g = e_{-\sigma} + e_{\mu}a_{+}, \quad \mu, \sigma > 0, \quad a_{+} \in H_{\infty}^{+}, \quad (4.29)$$

where $\mu + \sigma \geq \lambda$. Then the Toeplitz operator T_G is invertible if (and only if, provided that $G \in \mathcal{B}^{2 \times 2}$)

$$\mu + \sigma = \lambda \text{ and } \inf_{\mathbb{C}^{+} + i\varepsilon} |a_{+}| > 0 \text{ for some } \varepsilon > 0, \quad (4.30)$$

and T_G is not semi-Fredholm if $\mu + \sigma > \lambda$.

Proof. Condition (4.29) implies that $g \in S_{\lambda,1}$ with $\beta = \sigma$, $\nu = \lambda - \sigma$, and a solution to (1.7) is given by

$$\phi_{+} = (e_{\sigma}, -e_{\mu+\sigma-\lambda}a_{+}), \quad \phi_{-} = (e_{\sigma-\lambda}, 1).$$

Clearly, $\phi_{-} \in CP^{-}$, while $\phi_{+} \in CP^{+}$ if and only if (4.30) holds. The part of the statement pertinent to the case $\lambda = \sigma + \mu$ now follows from Theorems 2.10, 2.11.

For $\mu + \sigma > \lambda$, following the proof of [5, Theorem 5.3] observe that $\frac{1-e_{-\gamma}(z)}{z}\phi_{\pm}(z)$ deliver a solution to (1.7) in L_p , for any γ between 0 and $\min\{\sigma, \mu + \sigma - \lambda\}$. Thus, the operator T_G has an infinite dimensional kernel in $(H_p^{+})^2$ for any $p \in (1, \infty)$.

Denote by G^{-T} the transposed of G^{-1} . A direct computation shows that for the matrix under consideration, due to its algebraic structure,

$$G^{-T} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} G \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}. \quad (4.31)$$

Therefore, the operator $T_{G^{-T}}$ also has an infinite dimensional kernel. But this means (see, e.g., [12, Section 3.1]) that the cokernel of T_G is infinite dimensional. Therefore, the operator T_G is not even semi-Fredholm on $(H_p^{+})^2$, $1 < p < \infty$. \square

Theorem 4.9. *Let, as in Theorem 4.8, (4.1) and (4.29) hold, but now with*

$$\mu \in \left[\frac{\lambda}{N} - \sigma, \frac{\lambda}{N} - \frac{N-1}{N}\sigma \right] \text{ and } e_{-\frac{\mu}{N-1}}a_{+} \in H_{\infty}^{-}$$

for some integer $N > 1$. Then T_G is invertible if (and only if, for $G \in \mathcal{B}^{2 \times 2}$) for some $\varepsilon > 0$ one of the following three conditions holds:

$$\sigma + \mu = \frac{\lambda}{N} \text{ and } \inf_{\mathbb{C}^{+} + i\varepsilon} |a_{+}| > 0, \quad (4.32)$$

or

$$\frac{N-1}{N}\sigma + \mu = \frac{\lambda}{N} \text{ and } \inf_{\mathbb{C}^{-} - i\varepsilon} \left| e_{-\frac{\mu}{N-1}}a_{+} \right| > 0, \quad (4.33)$$

or

$$\inf_{\mathbb{C}^{+} + i\varepsilon} |a_{+}| > 0, \text{ and } \inf_{\mathbb{C}^{-} - i\varepsilon} \left| e_{-\frac{\mu}{N-1}}a_{+} \right| > 0.$$

If, on the other hand,

$$\sigma + \mu > \frac{\lambda}{N} \text{ and } e_{\delta - \frac{\mu}{N-1}} a_+ \in H_\infty^- \quad (4.34)$$

or

$$\frac{N-1}{N}\sigma + \mu < \frac{\lambda}{N} \text{ and } e_{-\delta} a_+ \in H_\infty^+ \quad (4.35)$$

for some $\delta > 0$, then T_G is not even semi-Fredholm.

Proof. According to Lemma 4.7, $G \in \mathfrak{S}_{\lambda, N}$. Moreover, one can choose in (4.2) $\nu = \mu$, $\beta = \frac{\lambda}{N} - \mu$ and $a_- = e_{\frac{\lambda}{N} - \mu - \sigma}$. Then formulas (4.7), (4.8) yield the following solution to (1.7):

$$\begin{aligned} \phi_{1+} &= e_{\lambda - N(\mu + \sigma) + \sigma} \sum_{j=0}^{N-1} \left((-1)^j a_+^{N-1-j} e_{(N-1-j)(\mu + \sigma)} \right), \\ \phi_{2+} &= -a_+^N, \\ \phi_{1-} &= \sum_{j=0}^{N-1} \left((-1)^j (e_{-\frac{\mu}{N-1}} a_+)^{N-1-j} e_{-j(\frac{N}{N-1}\mu + \sigma)} \right), \\ \phi_{2-} &= (-1)^{N-1} e_{\lambda - N(\sigma + \mu)}. \end{aligned}$$

Clearly, $(\phi_{1-}, \phi_{2-}) \in CP^-$ if and only if the first condition in (4.32) or the second condition in (4.33) holds. Similarly, $(\phi_{1+}, \phi_{2+}) \in CP^+$ is equivalent to the first condition in (4.33) or the second condition in (4.32). Since the first conditions in (4.32), (4.33) cannot hold simultaneously, the statement regarding the invertibility of T_G now follows from Theorems 2.10, 2.11.

If (4.34) or (4.35) holds, then $\phi_- = e_{-\tilde{\delta}} \tilde{\phi}_-$ or $\phi_+ = e_{\tilde{\delta}} \tilde{\phi}_+$ with $\tilde{\delta} > 0$, $\tilde{\phi}_\pm \in (H_\infty^\pm)^2$, respectively. It follows that the kernel of T_G is infinite dimensional, as in the proof of Theorem 4.8. Using (4.31), we in the same manner derive that the cokernel of T_G also is infinite dimensional. So, T_G is not semi-Fredholm. \square

5. AP matrix functions with a spectral gap around zero

The results of the previous section take a particular and, in some sense, more explicit form when considered in the almost periodic setting. The first natural question is, which functions $g \in AP$ belong to $S_{\lambda, N}$ for some $N \in \mathbb{N}$, with $a_\pm \in AP^\pm$ in (4.2).

According to Remark 4.1, we may have $0 \in \Omega(g)$ only if $N = 1$ and, in addition, $g = a_- + a_+ e_\lambda$ with $0 \in \Omega(a_-)$ or $g = a_- e_{-\lambda} + a_+$ with $0 \in \Omega(a_+)$. In either case the operator T_G is invertible, as can be deduced from the so called one sided case, see [3, Section 14.1]. The easiest way to see that directly, however, is by observing that problem (1.7) has a solution on CP^\pm : $\phi_+ = (1, -a_+)$, $\phi_- = (e_{-\lambda}, a_-)$ in the first case, $\phi_+ = (e_\lambda, -a_+)$, $\phi_- = (1, a_-)$ in the second.

Therefore, in what follows we restrict ourselves to the case $0 \notin \Omega(g)$. Then

$$g = g_- + g_+ \text{ with } g_{\pm} \in AP^{\pm}, \quad 0 \notin \Omega(g_{\pm}) \quad (5.1)$$

with g_{\pm} uniquely defined by g . Comparing with (4.2), we have

$$g_+ = a_+ e_{\nu}, \quad g_- = a_- e_{-\beta}. \quad (5.2)$$

Let

$$\eta_{1-} = -\sup \Omega(g_-), \quad \eta_{2-} = -\inf \Omega(g_-), \quad (5.3)$$

$$\eta_{1+} = \inf \Omega(g_+), \quad \eta_{2+} = \sup \Omega(g_+). \quad (5.4)$$

Here $\Omega(g_+), -\Omega(g_-)$ are thought of as subsets of \mathbb{R}_+ (possibly empty), so that $\eta_{1\pm}, \eta_{2\pm} \in [0, +\infty] \cup \{-\infty\}$.

Theorem 5.1. *Let g be given by (5.1). Then*

- (i) $g \in S_{\lambda,1}$ if and only if $\eta_{1+} + \eta_{1-} \geq \lambda$;
- (ii) $g \in S_{\lambda,N}$ with $N > 1$ if and only if

$$N = \left\lceil \frac{\lambda}{\eta_{1-} + \eta_{1+}} \right\rceil, \quad (5.5)$$

while

$$\eta_{1-} \geq \frac{N-1}{N} \eta_{2-}, \quad \eta_{1+} \geq \frac{N-1}{N} \eta_{2+}, \quad \eta_{2+} + \eta_{2-} \leq \frac{\lambda}{N-1}. \quad (5.6)$$

Under these conditions, any ν satisfying

$$M := \max \left\{ \frac{\lambda}{N} - \eta_{1-}, \frac{N-1}{N} \eta_{2+} \right\} \leq \nu \leq \min \left\{ \eta_{1+}, \frac{\lambda - (N-1)\eta_{2-}}{N} \right\} =: m \quad (5.7)$$

and

$$a_+ = g_+ e_{-\nu}, \quad \beta = \frac{\lambda}{N} - \nu, \quad a_- = g_- e_{\beta} \quad (5.8)$$

deliver a representation (4.2).

Proof. (i) If $g \in S_{\lambda,1}$, then from (5.2) with $\nu + \beta = \lambda$ it follows that $\eta_{1+} + \eta_{1-} \geq \lambda$. Conversely, setting $a_{\pm} = 0$ if $g_{\pm} = 0$, $a_+ = g_+ e_{-\eta_{1+}}$, $a_- = g_- e_{\lambda - \eta_{1+}}$ if $g_+ \neq 0$, and $a_+ = g_+ e_{-\lambda + \eta_{1-}}$, $a_- = g_- e_{\eta_{1-}}$ if $g_- \neq 0$, we can write g as in (4.2) with $\nu + \beta = \lambda$, so that $g \in S_{\lambda,1}$.

(ii) *Necessity.* Formulas for a_{\pm} in (5.8) follow from the uniqueness of g_{\pm} in the representation (5.1). The condition $a_{\pm} \in H_{\infty}^{\pm}$ is therefore equivalent to

$$\beta \leq \eta_{1-}, \quad \nu \leq \eta_{1+}. \quad (5.9)$$

Conditions (4.6), in their turn, are equivalent to

$$\beta \geq \frac{N-1}{N} \eta_{2-}, \quad \nu \geq \frac{N-1}{N} \eta_{2+}. \quad (5.10)$$

Comparing the respective inequalities in (5.9) and (5.10) shows the necessity of the first two conditions in (5.6). To obtain the third condition there, just add the two inequalities in (5.10):

$$\beta + \nu \geq \frac{N-1}{N}(\eta_{2+} + \eta_{2-}),$$

and compare the result with (4.5).

On the other hand, adding the inequalities in (5.9) yields, once again with the use of (4.5),

$$\frac{\lambda}{N} = \beta + \nu \leq \eta_{1+} + \eta_{1-}.$$

So,

$$\frac{\lambda}{\eta_{1+} + \eta_{1-}} \leq N \leq 1 + \frac{\lambda}{\eta_{2+} + \eta_{2-}}. \quad (5.11)$$

If at least one of the inequalities $\eta_{2\pm} > \eta_{1\pm}$ holds, the difference between the right- and left-hand sides of the inequalities (5.11) is strictly less than 1, and therefore an integer N is defined by (5.11) uniquely, in accordance with (5.5). Otherwise, $\eta_{1\pm} = \eta_{2\pm}$, which means that $g = c_1 e_{\eta_{1-}} + c_2 e_{\eta_{1+}}$ with $c_1, c_2 \in \mathbb{C} \setminus \{0\}$. Since by definition N is the smallest possible number satisfying (4.4) with ν, β such that (4.2) holds, we arrive again at (5.5).

Sufficiency. Let (5.6) hold for N defined by (5.5). Then m, M defined in (5.7) satisfy $M \leq m$, so that ν may indeed be chosen as in (5.7). With such ν , and a_{\pm} defined by (5.8), we have (4.2), (4.5), and (4.6). \square

The results of Theorem 4.5 and Corollary 4.6, combined with Theorem 5.1, yield the following.

Theorem 5.2. *Let $g \in S_{\lambda, N}$ be written as (5.1), and let $\eta_{j\pm}$ ($j = 1, 2$) be defined by (5.3)–(5.4). Then the Toeplitz operator T_G with symbol G given by (4.1) is invertible if (and, for $g \in APW$, only if) one of the following conditions holds:*

(i): $N = 1$ and

$$\eta_{1+} \in \Omega(g_+), \quad -\eta_{1-} \in \Omega(g_-), \quad \eta_{1+} + \eta_{1-} = \lambda; \quad (5.12)$$

(ii): $N > 1$ and

$$\eta_{1+} \in \Omega(g_+), \quad -\eta_{1-} \in \Omega(g_-), \quad \eta_{1+} + \eta_{1-} = \frac{\lambda}{N}; \quad (5.13)$$

(iii): $N > 1$ and

$$\eta_{1+}, \eta_{2+} \in \Omega(g_+), \quad \eta_{2+} = \frac{N}{N-1} \eta_{1+}; \quad (5.14)$$

(iv): $N > 1$ and

$$-\eta_{1-}, -\eta_{2-} \in \Omega(g_-), \quad \eta_{2-} = \frac{N}{N-1} \eta_{1-}; \quad (5.15)$$

(v): $N > 1$ and

$$\eta_{2+} \in \Omega(g_+), \quad -\eta_{2-} \in \Omega(g_-), \quad \eta_{2+} + \eta_{2-} = \frac{\lambda}{N-1}; \quad (5.16)$$

and, whenever $N > 1$,

$$\inf_S (|g_+| + |g_-|) > 0 \text{ for any strip } S \text{ of the form (4.11)}. \quad (5.17)$$

Proof. For $N = 1$, (5.12) is equivalent to (4.16).

For $N > 1$, setting

$$a_- = e_{\frac{\lambda}{N}-\nu} g_- \quad \text{and} \quad a_+ = e_{-\nu} g_+ \quad (5.18)$$

where $\beta = \frac{\lambda}{N} - \nu$, we deduce from (4.6) that

$$b_- = e_{-\frac{N\nu}{N-1}} g_+ \quad \text{and} \quad b_+ = e_{\frac{\lambda-N\nu}{N-1}} g_-. \quad (5.19)$$

Hence

$$\begin{aligned} M(a_+) \neq 0 & \quad \text{if and only if} \quad \eta_{1+} = \nu \in \Omega(g_+), \\ M(b_+) \neq 0 & \quad \text{if and only if} \quad -\eta_{2-} = -\frac{\lambda-N\nu}{N-1} \in \Omega(g_-), \\ M(a_-) \neq 0 & \quad \text{if and only if} \quad -\eta_{1-} = \nu - \frac{\lambda}{N} \in \Omega(g_-), \\ M(b_-) \neq 0 & \quad \text{if and only if} \quad \eta_{2+} = -\frac{N\nu}{N-1} \in \Omega(g_+). \end{aligned}$$

Thus, the first inequality in (4.17) holds if and only if either $\eta_{1+} = \nu \in \Omega(g_+)$ or $-\eta_{2-} = -\frac{\lambda-N\nu}{N-1} \in \Omega(g_-)$, and the second inequality in (4.17) holds if and only if either $-\eta_{1-} = \nu - \frac{\lambda}{N} \in \Omega(g_-)$ or $\eta_{2+} = -\frac{N\nu}{N-1} \in \Omega(g_+)$.

Taking now $\eta_{1+} = \nu \in \Omega(g_+)$ and $-\eta_{1-} = \nu - \frac{\lambda}{N} \in \Omega(g_-)$, we get the equivalence of (4.17) and (5.13); taking $\eta_{1+} = \nu \in \Omega(g_+)$ and $\eta_{2+} = -\frac{N\nu}{N-1} \in \Omega(g_+)$, we get the equivalence of (4.17) and (5.14); taking $-\eta_{2-} = -\frac{\lambda-N\nu}{N-1} \in \Omega(g_-)$ and $-\eta_{1-} = \nu - \frac{\lambda}{N} \in \Omega(g_-)$, we get the equivalence of (4.17) and (5.15); taking $-\eta_{2-} = -\frac{\lambda-N\nu}{N-1} \in \Omega(g_-)$ and $\eta_{2+} = -\frac{N\nu}{N-1} \in \Omega(g_+)$, we get the equivalence of (4.17) and (5.16). Thus, we see that (4.17) is equivalent to one of the conditions (ii)–(v) of the theorem being satisfied.

The result now follows from Theorem 4.5 and Corollary 4.6 and the second equivalence in (4.15). \square

From (5.7) it follows that in the case (ii) we have $\nu = \frac{\lambda}{N} - \eta_{1-} = \eta_{1+}$ and therefore

$$\lambda \geq \max\{N\eta_{1-} + (N-1)\eta_{2+}, N\eta_{1+} + (N-1)\eta_{2-}\},$$

in the case (iii) we have $\nu = \frac{N-1}{N} \eta_{2+} = \eta_{1+}$ so that

$$N\eta_{1+} + (N-1)\eta_{2-} \leq \lambda \leq N\eta_{1-} + (N-1)\eta_{2+},$$

in the case (iv) we have $\nu = \frac{\lambda}{N} - \eta_{1-} = \frac{\lambda}{N} - \frac{N-1}{N} \eta_{2-}$ and therefore

$$N\eta_{1-} + (N-1)\eta_{2+} \leq \lambda \leq N\eta_{1+} + (N-1)\eta_{2-},$$

in the case (v) we have $\nu = \frac{N-1}{N} \eta_{2+} = \frac{\lambda}{N} - \frac{N-1}{N} \eta_{2-}$ so that

$$\lambda \leq \min\{N\eta_{1-} + (N-1)\eta_{2+}, N\eta_{1+} + (N-1)\eta_{2-}\}.$$

We also note that if $\lambda = N\eta_{1-} + (N-1)\eta_{2+}$, then condition (5.14) is equivalent to

$$\eta_{1+}, \eta_{2+} \in \Omega(g_+), \quad \eta_{1+} + \eta_{1-} = \frac{\lambda}{N}; \quad (5.20)$$

while condition (5.15) is equivalent to

$$-\eta_{1-}, -\eta_{2-} \in \Omega(g_-), \quad \eta_{2+} + \eta_{2-} = \frac{\lambda}{N-1}. \quad (5.21)$$

If $\lambda = N\eta_{1+} + (N-1)\eta_{2-}$, then condition (5.14) is equivalent to (5.21), while condition (5.15) is equivalent to (5.20).

Observe that necessity of conditions (5.12)–(5.16) persists for $g \in AP$ without an additional restriction $g \in APW$. To see that, suppose that T_G is invertible in one of the cases (i)–(v) while the respective condition (5.12)–(5.16) fails. Approximate g by a function in APW with the same $\eta_{j\pm}$ and so close to g in the uniform norm that the respective Toeplitz operator is still invertible. This contradicts the necessity of (5.12)–(5.16) in the APW case.

It is not clear, however, whether the condition (5.17) remains necessary in the AP setting.

Remark 5.3. *Part (i) of Theorem 5.2 means that, for T_G to be invertible in the case when the length of the spectral gap of g around zero is at least λ , it in fact must equal λ and, moreover, both endpoints of the spectral gap must belong to $\Omega(g)$. In contrast with this, for $N > 1$ according to parts (ii)–(v) T_G can be invertible when one (or both) of the endpoints of the spectral gap around zero is missing from $\Omega(g)$, and the length of this spectral gap can be greater than λ/N .*

For $g \in APW$ Theorem 5.2 delivers the invertibility criterion of T_G , and thus a necessary and sufficient condition for G to admit a canonical APW factorization. Using Theorem 3.8, however, will allow us to tackle the non-canonical AP factorability of G as well.

We assume from now on that $g \in APW$ is given by (5.1), so that in fact $g_{\pm} \in APW^{\pm}$, and that $g \in S_{\lambda,N}$ as described by Theorem 5.1.

In the notation of this theorem, for $N = 1$ we have $\eta_{1+} + \eta_{1-} \geq \lambda$ — the so called *big gap* case, — and a solution to (1.7) is given by

$$\phi_+ = (e_{\lambda-\nu}, -e_{-\nu+\eta_{1+}}\tilde{g}_+), \quad (5.22)$$

$$\phi_- = (e_{-\nu}, e_{\lambda-\nu-\eta_{1-}}\tilde{g}_-), \quad (5.23)$$

where

$$\tilde{g}_+ = e_{-\eta_{1+}}g_+ \quad (0 = \inf \Omega(\tilde{g}_+)), \quad (5.24)$$

$$\tilde{g}_- = e_{\eta_{1-}}g_- \quad (0 = \sup \Omega(\tilde{g}_-)), \quad (5.25)$$

$$\max\{0, \lambda - \eta_{1-}\} \leq \nu \leq \min\{\eta_{1+}, \lambda\}. \quad (5.26)$$

Knowing these solutions and using Theorem 3.4 with $f \in APW^+$ as in (3.6), we will be able to complete the consideration of AP factorability in the big gap case.

It was shown earlier (see [3, Chapter 14], [7, Theorem 2.2]) that G is APW factorable if, in addition to the big gap requirement $\eta_{1+} + \eta_{1-} \geq \lambda$, also

$$\eta_{1+} \in \Omega(g_+) \text{ or } \eta_{1+} \geq \lambda, \quad -\eta_{1-} \in \Omega(g_-) \text{ or } \eta_{1-} \geq \lambda. \quad (5.27)$$

However, the AP factorability of G if $\lambda > \eta_{1+} \notin \Omega(g_+)$ or $\lambda > \eta_{1-} \notin -\Omega(g_-)$ remained unsettled. As the next theorem shows, in these cases G does not have an AP factorization.

Theorem 5.4. *Let $g \in APW$ be given by (5.1), with $\eta_{1\pm}$ defined by (5.3), (5.4) and satisfying $\eta_{1+} + \eta_{1-} \geq \lambda$. Then the matrix function (4.1) is AP factorable if and only if (5.27) holds. In this case G actually admits an APW factorization and its partial indices are $\pm\mu$ with*

$$\mu = \min\{\lambda, \eta_{1+}, \eta_{1-}, \eta_{1+} + \eta_{1-} - \lambda\}. \quad (5.28)$$

In particular, the factorization is canonical if and only if $\eta_{1+} = 0$ or $\eta_{1-} = 0$ or $\eta_{1+} + \eta_{1-} = \lambda$.

Proof. Sufficiency. Although it was established earlier, we give here a (much) shorter and self-contained proof, based on the results of Theorem 3.8. Namely, if (5.27) is satisfied, then (5.22)–(5.26) hold with $0 = \min \Omega(\tilde{g}_+) = \max \Omega(\tilde{g}_-)$. Writing

$$\begin{aligned} \phi_+ &= e_{\mu_1} \tilde{\psi}_+ \quad \text{with} \quad \mu_1 = \min\{\lambda - \nu, -\nu + \eta_{1+}\}, \\ \phi_- &= e_{-\mu_2} \tilde{\psi}_- \quad \text{with} \quad \mu_2 = \min\{\nu, \eta_{1-} + \nu - \lambda\}, \end{aligned}$$

we see that $\tilde{\psi}_{\pm} \in APW^{\pm} \cap CP^{\pm}$ and

$$Ge_{\mu_1+\mu_2} \tilde{\psi}_+ = \tilde{\psi}_-,$$

so that, according to Theorem 3.8, G admits an APW factorization with partial indices $\pm\mu$ where

$$\mu = \mu_1 + \mu_2 = \min\{\lambda, \eta_{1+}, \eta_{1-}, \eta_{1+} + \eta_{1-} - \lambda\}$$

(as can be checked straightforwardly).

Necessity. Suppose that $\Omega(g_+) \not\ni \eta_{1+} < \lambda$; the case $-\Omega(g_-) \not\ni \eta_{1-} < \lambda$ can be treated analogously. Then a solution to (1.7) with $\phi_{\pm} \in (APW^{\pm})^2$ is given by (5.22)–(5.26).

It follows from these formulas that $\phi_{2+} = -e_{-\nu+\eta_{1+}} \tilde{g}_+$, where $-\nu + \eta_{1+} \geq 0$ due to (5.26). On the other hand, $0 \notin \Omega(\tilde{g}_+)$ because $\eta_{1+} \notin \Omega(g_+)$. Therefore, for any $\varepsilon > 0$ and $\nu = \eta_{1+}$ there is $y_{\varepsilon} \in \mathbb{R}^+$ such that

$$\inf_{\mathbb{C}^+ + iy_{\varepsilon}} |\phi_{2+}| = \inf_{\mathbb{C}^+ + iy_{\varepsilon}} |e_{-\nu+\eta_{1+}} \tilde{g}_+| < \varepsilon$$

and

$$\inf_{\mathbb{C}^+ + iy_\varepsilon} |\phi_{1+}| = \inf_{\mathbb{C}^+ + iy_\varepsilon} |e_{\lambda-\nu}| < \varepsilon.$$

Thus $\phi_+ = (\phi_{1+}, \phi_{2+}) \notin CP^+$ and we conclude from Theorem 3.7 that G cannot have a canonical AP factorization.

Now, if G admits a non-canonical factorization, which must have partial AP indices $\pm\mu$ with $\mu > 0$, then according to Corollary 3.4 we have (3.6) with $f \in AP^+$, $\Omega(f) \subset [0, \mu]$. Denoting $g_1^\pm = (g_{11}^\pm, g_{21}^\pm)$, and considering in particular the first component of ϕ_+ , we thus have from (5.22):

$$e_{\lambda-\nu} = fg_{11}^+. \quad (5.29)$$

In addition, from the factorization it follows directly that

$$e_{-\lambda+\mu}g_{11}^+ = g_{11}^-.$$

Consequently, the Bohr-Fourier spectrum of g_{11}^+ also is bounded, and (5.29) therefore holds everywhere in \mathbb{C} . In particular, f and g_{11}^+ do not vanish in \mathbb{C} . But then (see [9, Lemma 3.2] or [11, p. 371]) $\Omega(f)$, $\Omega(g_{11}^+)$ must each contain the maximum and the minimum element, which implies that

$$\max \Omega(f) + \max \Omega(g_{11}^+), \min \Omega(f) + \min \Omega(g_{11}^+) \in \Omega(fg_{11}^+) = \{\lambda - \nu\}.$$

We conclude that $\min \Omega(f) = \max \Omega(f)$ and thus $f = e_\gamma$ for some $\gamma \in [0, \mu]$.

But then, from (5.22) and (3.6),

$$(g_{11}^+, g_{21}^+) = (e_{\lambda-\nu-\gamma}, e_{-\nu+\eta_{1+}-\gamma}\tilde{g}_+) \in CP^+,$$

which is impossible when $\Omega(g_+) \not\ni \eta_{1+} < \lambda$. Indeed, in this case $\lambda - \nu - \gamma > \eta_{1+} - \nu - \gamma \geq 0$ and $0 \notin \Omega(\tilde{g}_+)$.

Finally, the criterion for the AP factorization of G to be canonical, when it exists, follows immediately from formulas (5.28). \square

Remark 5.5. *The last statement of Theorem 3.8 implies that the construction in the proof of Theorem 5.4 delivers not only the partial AP indices but also a first column of G_+ and G_- . Namely, they may be chosen equal to $\tilde{\psi}_+$ and $\tilde{\psi}_-$, respectively.*

Now we move to the case $N > 1$.

Knowing a solution (4.20), (4.21) of (1.7) and using Theorem 3.8 (with $\det G \equiv 1$, and therefore $\delta = 0$), we can obtain sufficient conditions for AP factorability of $G \in \mathfrak{S}_{\lambda, N}$, $N > 1$.

Theorem 5.6. *Let $g \in APW$ be such that $g \in S_{\lambda, N}$, $N > 1$, as described in Theorem 5.1, with (5.17) satisfied. Then G admits an APW factorization with partial AP indices $\pm\mu$ where:*

(i): $\mu = N(\eta_{1+} + \eta_{1-}) - \lambda$ if

$$\eta_{1+} \in \Omega(g_+), \quad -\eta_{1-} \in \Omega(g_-) \quad (5.30)$$

and

$$\lambda \geq \max \{N\eta_{1+} + (N-1)\eta_{2-}, N\eta_{1-} + (N-1)\eta_{2+}\}; \quad (5.31)$$

(ii): $\mu = N\eta_{1+} - (N-1)\eta_{2+}$ if

$$\eta_{1+}, \eta_{2+} \in \Omega(g_+) \quad (5.32)$$

and

$$N\eta_{1+} + (N-1)\eta_{2-} \leq \lambda \leq N\eta_{1-} + (N-1)\eta_{2+}; \quad (5.33)$$

(iii): $\mu = N\eta_{1-} - (N-1)\eta_{2-}$ if

$$-\eta_{1-}, -\eta_{2-} \in \Omega(g_-) \quad (5.34)$$

and

$$N\eta_{1-} + (N-1)\eta_{2+} \leq \lambda \leq N\eta_{1+} + (N-1)\eta_{2-}; \quad (5.35)$$

(iv): $\mu = \lambda - (N-1)(\eta_{2+} + \eta_{2-})$ if

$$\eta_{2+} \in \Omega(g_+), \quad -\eta_{2-} \in \Omega(g_-) \quad (5.36)$$

and

$$\lambda \leq \min \{N\eta_{1+} + (N-1)\eta_{2-}, N\eta_{1-} + (N-1)\eta_{2+}\}. \quad (5.37)$$

Proof. Consider the solution to (1.7) given by (4.20), (4.21), with a_{\pm}, b_{\pm} as in (5.18), (5.19). Then we obtain

$$\begin{aligned} \phi_{1+} &= e_{\lambda-N\nu} \sum_{j=0}^{N-1} \left((-1)^j g_-^j g_+^{N-1-j} \right) \\ &= \sum_{j=0}^{N-1} \left((-1)^j e_{\lambda-N\nu-j\eta_{2-}+(N-1-j)\eta_{1+}} (e_{\eta_{2+}} g_-)^j (e_{-\eta_{1+}} g_+)^{N-1-j} \right) \\ &= e_{\lambda-N\nu-(N-1)\eta_{2-}} \tilde{\phi}_{1+}, \end{aligned}$$

with $\tilde{\phi}_{1+} \in APW^+$ where $\lambda-N\nu-(N-1)\eta_{2-} \geq 0$ due to (5.7) and $0 = \inf \Omega(\tilde{\phi}_{1+})$ ($= \min \Omega(\tilde{\phi}_{1+})$ if $-\eta_{2-} \in \Omega(g_-)$);

$$\phi_{2+} = -e_{-N\nu} g_+^N = -e_{-N\nu+N\eta_{1+}} (e_{-\eta_{1+}} g_+)^N = e_{-N\nu+N\eta_{1+}} \tilde{\phi}_{2+},$$

with $\tilde{\phi}_{2+} \in APW^+$ where $-N\nu + N\eta_{1+} \geq 0$ due to (5.7) and $0 = \inf \Omega(\tilde{\phi}_{2+})$ ($= \min \Omega(\tilde{\phi}_{2+})$ if $\eta_{1+} \in \Omega(g_+)$);

$$\begin{aligned} \phi_{1-} &= e_{-N\nu} \sum_{j=0}^{N-1} \left((-1)^j g_-^j g_+^{N-1-j} \right) \\ &= \sum_{j=0}^{N-1} \left((-1)^j e_{-N\nu-j\eta_{1-}+(N-1-j)\eta_{2+}} (e_{\eta_{1-}} g_-)^j (e_{-\eta_{2+}} g_+)^{N-1-j} \right) \\ &= e_{-N\nu+(N-1)\eta_{2+}} \tilde{\phi}_{1-}, \end{aligned}$$

with $\tilde{\phi}_{1-} \in APW^-$ where $-N\nu + (N-1)\eta_{2+} \leq 0$ due to (5.7) and $0 = \sup \Omega(\tilde{\phi}_{1-})$ ($= \max \Omega(\tilde{\phi}_{1-})$ if $\eta_{2+} \in \Omega(g_+)$);

$$\phi_{2-} = (-1)^{N-1} e_{\lambda-N\nu} g_-^N = (-1)^{N-1} e_{\lambda-N\nu-N\eta_{1-}} (e_{\eta_{1-}} g_-)^N = e_{\lambda-N\nu-N\eta_{1-}} \tilde{\phi}_{2-},$$

with $\tilde{\phi}_{2-} \in APW^-$ where $\lambda - N\nu + N\eta_{1-} \geq 0$ due to (5.7) and $0 = \sup \Omega(\tilde{\phi}_{2-})$ ($= \max \Omega(\tilde{\phi}_{2-})$ if $-\eta_{1-} \in \Omega(g_-)$).

Hence,

$$G \begin{bmatrix} e_{\lambda - N\nu - (N-1)\eta_{2-}} \tilde{\phi}_{1+} \\ e_{-N\nu + N\eta_{1+}} \tilde{\phi}_{2+} \end{bmatrix} = \begin{bmatrix} e_{-N\nu + (N-1)\eta_{2+}} \tilde{\phi}_{1-} \\ e_{\lambda - N\nu - N\eta_{1-}} \tilde{\phi}_{2-} \end{bmatrix}. \quad (5.38)$$

Setting now $\phi_+ = e_{\mu_1} \tilde{\psi}_+$ and $\phi_- = e_{-\mu_2} \psi_-$ where

$$\begin{aligned} \mu_1 &= -N\nu + \min \{ \lambda - (N-1)\eta_{2-}, N\eta_{1+} \} \geq 0, \\ \mu_2 &= N\nu + \min \{ -(N-1)\eta_{2+}, N\eta_{1-} - \lambda \} \geq 0, \end{aligned}$$

we infer from (5.38) that $G\psi_+ = \psi_-$, with $\psi_+ = e_{\mu} \tilde{\psi}_+$ and

$$\begin{aligned} \mu &= \mu_1 + \mu_2 = \min \{ \lambda - (N-1)\eta_{2-}, N\eta_{1+} \} + \min \{ -(N-1)\eta_{2+}, N\eta_{1-} - \lambda \} \\ &= \min \{ N(\eta_{1+} + \eta_{1-}) - \lambda, N\eta_{1+} - (N-1)\eta_{2+}, \\ &\quad N\eta_{1-} - (N-1)\eta_{2-}, \lambda - (N-1)(\eta_{2+} + \eta_{2-}) \}. \end{aligned} \quad (5.39)$$

We consider separately the cases (i)-(iv).

(i) If (5.30) and (5.31) hold, then $\mu = N(\eta_{1+} + \eta_{1-}) - \lambda$ due to (5.39) and

$$\tilde{\psi}_+ = \begin{bmatrix} e_{\lambda - N\eta_{1+} - (N-1)\eta_{2-}} \tilde{\phi}_{1+} \\ \tilde{\phi}_{2+} \end{bmatrix}, \quad \psi_- = \begin{bmatrix} e_{-\lambda + N\eta_{1-} + (N-1)\eta_{2+}} \tilde{\phi}_{1-} \\ \tilde{\phi}_{2-} \end{bmatrix}$$

where $M(\tilde{\phi}_{2+}) \neq 0$ if and only $\eta_{1+} \in \Omega(g_+)$, and $M(\tilde{\phi}_{2-}) \neq 0$ if and only $-\eta_{1-} \in \Omega(g_-)$. Hence, by (5.30), $\tilde{\psi}_+ = e_{-\mu} \psi_+ \in CP^+$ and $\psi_- \in CP^-$. The result now follows from Theorem 3.8.

(ii) If (5.32) and (5.33) hold, then $\mu = N\eta_{1+} - (N-1)\eta_{2+}$ due to (5.39) and

$$\tilde{\psi}_+ = \begin{bmatrix} e_{\lambda - N\eta_{1+} - (N-1)\eta_{2-}} \tilde{\phi}_{1+} \\ \tilde{\phi}_{2+} \end{bmatrix}, \quad \psi_- = \begin{bmatrix} \tilde{\phi}_{1-} \\ e_{\lambda - N\eta_{1-} - (N-1)\eta_{2+}} \tilde{\phi}_{2-} \end{bmatrix}$$

where $M(\tilde{\phi}_{2+}) \neq 0$ if and only $\eta_{1+} \in \Omega(g_+)$, and $M(\tilde{\phi}_{1-}) \neq 0$ if and only $\eta_{2+} \in \Omega(g_+)$. Hence, by (5.32), $\tilde{\psi}_+ = e_{-\mu} \psi_+ \in CP^+$ and $\psi_- \in CP^-$. The result now follows from Theorem 3.8.

(iii) If (5.34) and (5.35) hold, then $\mu = N\eta_{1-} - (N-1)\eta_{2-}$ due to (5.39) and

$$\tilde{\psi}_+ = \begin{bmatrix} \tilde{\phi}_{1+} \\ e_{-\lambda + N\eta_{1+} + (N-1)\eta_{2-}} \tilde{\phi}_{2+} \end{bmatrix}, \quad \psi_- = \begin{bmatrix} e_{-\lambda + N\eta_{1-} + (N-1)\eta_{2+}} \tilde{\phi}_{1-} \\ \tilde{\phi}_{2-} \end{bmatrix}$$

where $M(\tilde{\phi}_{1+}) \neq 0$ if and only $-\eta_{2-} \in \Omega(g_-)$, and $M(\tilde{\phi}_{2-}) \neq 0$ if and only $-\eta_{1-} \in \Omega(g_-)$. Hence, by (5.34), $\tilde{\psi}_+ = e_{-\mu} \psi_+ \in CP^+$ and $\psi_- \in CP^-$. The result now follows from Theorem 3.8.

(iv) If (5.36) and (5.37) hold, then $\mu = \lambda - (N-1)(\eta_{2+} + \eta_{2-})$ due to (5.39) and

$$\tilde{\psi}_+ = \begin{bmatrix} \tilde{\phi}_{1+} \\ e^{-\lambda + N\eta_{1+} + (N-1)\eta_{2-}} \tilde{\phi}_{2+} \end{bmatrix}, \quad \psi_- = \begin{bmatrix} \tilde{\phi}_{1-} \\ e^{\lambda - N\eta_{1-} - (N-1)\eta_{2+}} \tilde{\phi}_{2-} \end{bmatrix}$$

where $M(\tilde{\phi}_{1+}) \neq 0$ if and only $-\eta_{2-} \in \Omega(g_-)$, and $M(\tilde{\phi}_{1-}) \neq 0$ if and only $\eta_{2+} \in \Omega(g_+)$. Hence, by (5.36), $\tilde{\psi}_+ = e_{-\mu}\psi_+ \in CP^+$ and $\psi_- \in CP^-$. The result again follows from Theorem 3.8. \square

Remark 5.7. If $\lambda = N\eta_{1+} + (N-1)\eta_{2-} = N\eta_{1-} + (N-1)\eta_{2+}$, then all the numbers

$$\begin{aligned} N(\eta_{1+} + \eta_{1-}) - \lambda, \quad N\eta_{1+} - (N-1)\eta_{2+}, \\ N\eta_{1-} - (N-1)\eta_{2-}, \quad \lambda - (N-1)(\eta_{2+} + \eta_{2-}) \end{aligned}$$

coincide, and therefore μ in Theorem 5.6 is equal to their common value. Analogously, if $\lambda = N\eta_{1+} + (N-1)\eta_{2-}$, then

$$\begin{aligned} N(\eta_{1+} + \eta_{1-}) - \lambda &= N\eta_{1-} - (N-1)\eta_{2-}, \\ \lambda - (N-1)(\eta_{2+} + \eta_{2-}) &= N\eta_{1+} - (N-1)\eta_{2+}, \end{aligned}$$

and if $\lambda = N\eta_{1-} + (N-1)\eta_{2+}$, then

$$\begin{aligned} N(\eta_{1+} + \eta_{1-}) - \lambda &= N\eta_{1+} - (N-1)\eta_{2+}, \\ \lambda - (N-1)(\eta_{2+} + \eta_{2-}) &= N\eta_{1-} - (N-1)\eta_{2-}. \end{aligned}$$

Hence, in the latter two cases $\mu = \min \{N(\eta_{1+} + \eta_{1-}) - \lambda, \lambda - (N-1)(\eta_{2+} + \eta_{2-})\}$.

Remark 5.8. The main difficulty in applying Theorem 5.6 lies in verifying whether or not condition (5.17) holds. In this regard, Theorems 3.1 and 3.4 of [4] may be helpful. Also, as was mentioned before, (5.17) holds if a_+ or a_- is a single exponential. A class of matrix functions with such a_{\pm} was studied in [7], where the APW factorization of G was explicitly obtained. Naturally, conclusions of [7] match those that can be obtained by applying Theorem 5.6 to the same class. Furthermore, combining Corollary 3.4 and Theorem 3.5 of the present paper with the APW factorization obtained in [7], it is possible to characterize completely the solutions of (1.7) in that case.

Below we give examples of two cases in which condition (5.17) is also not hard to verify.

Example 5.9. Let the off-diagonal entry $g \in S_{\lambda, N}$ of the matrix (4.1) be given by

$$g = c_{-2}e_{-\eta_{2-}} + c_{-1}e_{-\eta_{1-}} + g_+$$

with $c_{-2}, c_{-1} \in \mathbb{C}$, $0 \leq \eta_{1-} < \eta_{2-}$ and $g_+ \in APW^+$ with Bohr-Fourier spectrum containing its maximum and minimum points η_{j+} , $j = 1, 2$.

If $N = 1$, which happens in particular if $c_{-1} = c_{-2} = 0$, then G is APW factorable with partial AP indices given by Theorem 5.4.

If $N > 1$, then it follows from Theorem 5.6 that G admits an *APW* factorization with partial *AP* indices $\pm\mu$, where

$$\mu = \begin{cases} N(\eta_{1+} + \eta_{1-}) - \lambda & \text{if } \lambda \geq \max\{N\eta_{1+} + (N-1)\eta_{2-}, N\eta_{1-} + (N-1)\eta_{2+}\}, \\ N\eta_{1+} - (N-1)\eta_{2+} & \text{if } N\eta_{1+} + (N-1)\eta_{2-} \leq \lambda \leq N\eta_{1-} + (N-1)\eta_{2+}, \\ N\eta_{1-} - (N-1)\eta_{2-} & \text{if } N\eta_{1-} + (N-1)\eta_{2+} \leq \lambda \leq N\eta_{1+} + (N-1)\eta_{2-}, \\ \lambda - (N-1)(\eta_{2+} + \eta_{2-}) & \text{if } \lambda \leq \min\{N\eta_{1+} + (N-1)\eta_{2-}, N\eta_{1-} + (N-1)\eta_{2+}\}, \end{cases}$$

whenever (5.17) holds. Moreover, the expressions given in the proof of Theorem 5.6 for $\phi_{1\pm}, \phi_{2\pm}$ in each case also provide, by using Theorem 3.8, one column for the factors G_{\pm} in an *APW* factorization of G .

In its turn, condition (5.17) is satisfied if and only if one of the coefficients c_{-1}, c_{-2} is zero or (if $c_{-1}c_{-2} \neq 0$)

$$\inf_{k \in \mathbb{Z}} |g_+(z_k)| > 0 \quad (5.40)$$

where $z_k, k \in \mathbb{Z}$, are the zeros of $g_- = c_{-2}e_{-\eta_{2-}} + c_{-1}e_{-\eta_{1-}}$, i.e.,

$$z_k = \frac{1}{\eta_{2-} - \eta_{1-}} \left(\arg \left(-\frac{c_{-2}}{c_{-1}} \right) + 2k\pi - i \log \left| \frac{c_{-2}}{c_{-1}} \right| \right).$$

If, in particular, g_+ also is a binomial, i.e.,

$$g_+ = c_1 e_{\eta_{1+}} + c_2 e_{\eta_{2+}} \quad (c_1, c_2 \in \mathbb{C}, 0 \leq \eta_{1+} < \eta_{2+})$$

then (5.40) is satisfied whenever one of the coefficients c_1, c_2 is zero. On the other hand, for $c_1, c_2 \neq 0$ condition (5.40) is equivalent to (cf. Lemma 3.3 in [2])

$$\left| \frac{c_1}{c_2} \right|^{\eta_{2-} - \eta_{1-}} \neq \left| \frac{c_{-2}}{c_{-1}} \right|^{\eta_{2+} - \eta_{1+}} \quad \text{if} \quad \frac{\eta_{2+} - \eta_{1+}}{\eta_{2-} - \eta_{1-}} \in \mathbb{R} \setminus \mathbb{Q};$$

and to

$$\left(-\frac{c_1}{c_2} \right)^q \neq \left(-\frac{c_{-2}}{c_{-1}} \right)^p \quad \text{if} \quad \frac{\eta_{2+} - \eta_{1+}}{\eta_{2-} - \eta_{1-}} = \frac{p}{q}, \quad \text{with} \quad p, q \in \mathbb{N} \quad \text{relatively prime.}$$

Example 5.10. Let $G \in \mathfrak{S}_{\lambda, N}$, $N > 1$, with the off-diagonal entry $g \in APW$ of the form $g = g_- + g_+$ where

$$g_+ = c_{\alpha} e_{\alpha} g_- + c_{\mu} e_{\mu},$$

$\alpha, \mu > 0$, $c_{\alpha}, c_{\mu} \in \mathbb{C}, c_{\mu} \neq 0$ and $\eta_{1\pm}, \eta_{2\pm} \in \Omega(g_{\pm})$ (see (5.3), (5.4)). It is easy to see that (5.17) holds. Theorem 5.6 implies therefore that G admits an *APW* factorization with partial *AP* indices as indicated in that theorem.

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